

# M. KONTSEVICH'S GRAPH COMPLEX AND THE GROTHENDIECK-TEICHMÜLLER LIE ALGEBRA

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**ABSTRACT.** We show that the zeroth cohomology of Kontsevich's graph complex is isomorphic to the Grothendieck-Teichmüller Lie algebra  $\mathfrak{grt}$ . The map is explicitly described. This result has applications to deformation quantization and Duflo theory. Also, it allows proving the freeness part of the Deligne-Drinfeld conjecture in some low orders. As a side result one obtains that the homotopy deformations of the Gerstenhaber operad are parameterized by  $\mathfrak{grt}$ . Finally, our methods give a second proof of a result of H. Furusho, stating that the pentagon equation for  $\mathfrak{grt}$ -elements implies the hexagon equation.

## 1. INTRODUCTION

There are two mysterious Lie algebras in the arena of Deformation Quantization. One of them is the Grothendieck-Teichmüller Lie algebra  $\mathfrak{grt}$ , which by its free and transitive action on the set of Drinfeld associators acts on D. Tamarkin's quantization. The Deligne-Drinfeld conjecture states that this Lie algebra is freely generated by generators  $\sigma_{2k+1}$ ,  $k = 1, 2, \dots$ . The other mysterious object is Kontsevich's graph complex  $\mathcal{GC}_2$ . It is naturally a differential graded Lie algebra, which acts on M. Kontsevich's deformation quantizations. It was previously unknown how to compute the cohomology of this Lie algebra. The main result of this paper is the following.

**Theorem 1.** *The zeroth cohomology of the Kontsevich graph complex, considered as Lie algebra, is isomorphic to the Grothendieck-Teichmüller Lie algebra.*

$$H^0(\mathcal{GC}_2) \cong \mathfrak{grt}$$

The complex  $\mathcal{GC}_2$  is also the complex governing “universal”  $L_\infty$ -derivations of the space of polyvector fields  $T_{\text{poly}}$ . Hence, less precisely, the result can be reformulated as saying that those universal derivations are exactly  $\mathfrak{grt}$ .

There are also other versions,  $\mathcal{GC}_n$ , of Kontsevich's graph complex, which are related to  $n$ -algebras. The precise relation is as follows.

**Theorem 2.** *Let  $e_n$  be the operad governing  $n$ -algebras,  $n = 2, 3, \dots$ . Let  $\text{Def}(e_n)$  be its deformation complex. Then*

$$H(\text{Def}(e_n)) \cong S^+(H(\mathcal{GC}_n)[-n-1] \oplus \mathbb{R}[-n-1] \oplus V_n)[n+1]$$

where

$$V_n = \bigoplus_{\substack{j \geq 1 \\ j \equiv 2n+1 \pmod{4}}} \mathbb{R}[-j]$$

and  $S^+(\dots)$  denotes the symmetric tensor product, without the zeroth term.

From the two previous results together with simple degree considerations one can extract the following corollaries.

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*Key words and phrases.* Formality, Deformation Quantization, Operads.

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**Corollary 1.** *The graph complex  $\mathrm{GC}_2$  has no cohomology in degrees smaller than  $-1$ .*

*Proof.* By degree reasons  $H^j(\mathrm{Def}(e_2)) = 0$  for  $j < -1$ .  $\square$

**Remark.** It is however not true that the cohomology of  $\mathrm{GC}_2$  is concentrated in degree 0. By computer experiments, I found a non-trivial class with 12 vertices and 19 edges, hence  $H^3(\mathrm{GC}_2) \neq 0$ .

**Corollary 2.** *The deformations of the Gerstenhaber operad up to homotopy are given by  $\mathrm{grt}$ , i.e., there are isomorphisms of Lie algebras*

$$H^0(\mathrm{Def}(e_2)) \cong \mathrm{grt} \cong H^0(\mathrm{GC}_2).$$

*Proof.* One compares the degree zero parts on both sides of the equation in the above theorem and uses the previous corollary.  $\square$

**Remark.** I learned very recently that a similar result has been obtained by B. Fresse [8].

**1.1. Structure of the paper.** We tried to keep the main body of this paper short, at the cost of a larger number of Appendices which contain some technical results. In section 2 we recall the definition of the graph complexes we work with. Section 3 contains a proof of Theorem 2. In section 4 some results about  $\mathrm{grt}$  are recalled or derived. The proof of the main Theorem 1 follows in section 5. Sections 7 to 8 contain some applications of this Theorem. Section 6 contains a largely independent “pedestrian” description of the map between  $\mathrm{grt}$  and  $\mathrm{GC}_2$ . This section is strictly speaking not necessary, but the author likes to have explicit formulas and algorithms. Finally the numerous appendices provide background for some constructions and notations used in the main text.

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## 2. GRAPH OPERADS AND COMPLEXES

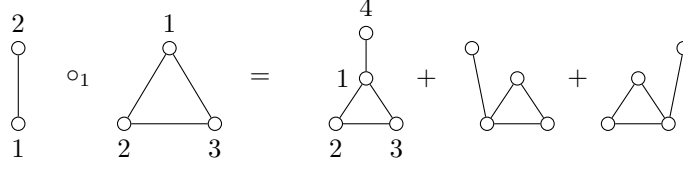
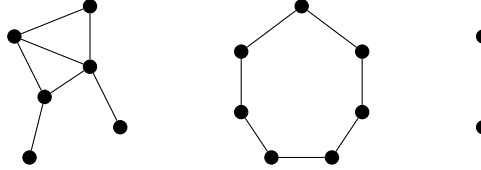
Several versions of graph complexes have been introduced by M. Kontsevich [14, 13].<sup>1</sup> Let us define them here in a combinatorial way. Let  $\mathbf{dgra}_{N,k}$  be the set of directed graphs with  $N$  labelled vertices and  $k$  ordered directed edges. Concretely, a graph in  $\mathbf{dgra}_{N,k}$  is given by an ordered set of  $k$  pairs  $(i, j)$ ,  $1 \leq i, j \leq N$ . There is a natural action of the permutation group  $P_k = S_k \times S_2^{\times k}$  on  $\mathbf{dgra}_{N,k}$  by permuting the order and flipping the direction of the edges. Denote by  $\mathrm{sgn}^{p,\pm}$  the one-dimensional representations of  $P_k$ , on which  $S_k$  acts trivially if  $p$  is even, by sign if  $p$  is odd, and on which each  $S_2$  acts trivially if “+” and by sign if “−”. Define graded vectorspaces of Graphs as follows.

$$\mathrm{Gra}_n^\pm(N) = \bigoplus_{k \geq 0} (\mathbb{R}\langle \mathbf{dgra}_{N,k} \rangle \otimes_{P_k} \mathrm{sgn}^{n-1,\pm}) [k(n-1)].$$

In words, we give each edge the degree  $1 - n$ , and introduce the appropriate signs. For “−” we additionally fix an orientation on the graph by prescribing directions on edges, identifying the graph with an edge direction flipped with minus the original graph. The space  $\mathrm{Gra}_n^\pm(N)$  naturally assemble to an operad  $\mathrm{Gra}_n^\pm$ . The  $S_N$  action

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<sup>1</sup>Strictly speaking, we consider the cohomological version here, while M. Kontsevich mainly considers the homological version. It does not matter much.

FIGURE 1. The operadic insertion in the operads  $\mathbf{Gra}_n^\pm$ .FIGURE 2. Three graphs in  $\mathbf{fGC}_{m,n}^\pm$ . The right hand graph is a Maurer-Cartan element.

is given simply by permuting the labels on the vertices. The insertions  $\Gamma_1 \circ_j \Gamma_2$  are given by “plugging in” the graph  $\Gamma_2$  at vertex  $j$  of graph  $\Gamma_1$  and reconnecting the edges incident to vertex  $j$  in  $\Gamma_1$  in all possible ways to vertices of  $\Gamma_2$ . See Figure 1. In the case  $n$  is even one needs to put an ordering (up to signed permutation) on the edges of the newly formed graphs. The natural choice is to consider the edges of  $\Gamma_1$  to stand on the left of those of  $\Gamma_2$ .

Next, we want to twist the operads  $\mathbf{Gra}_n^\pm$ , see Appendix G. First, one obtains a graded Lie algebra

$$\mathbf{fGC}_{m,n}^\pm = \text{Def}(\text{hoLie}_m \xrightarrow{0} \mathbf{Gra}_n^\pm)$$

of derivations of the zero map of operads. Elements of this Lie algebra are series of graphs as before, but with “unidentifiable” vertices. We will draw those vertices black in pictures, see Figure 2. In order to twist, we need a Maurer-Cartan element. We require that the graph with two vertices and one edge (see Figure 2) is such a Maurer-Cartan element. This has several consequences:

- (1) It must have degree 1. Hence we are forced to take  $m = n$ , i.e., consider  $\mathbf{fGC}_{n,n}^\pm$ .
- (2) It should not be zero by symmetry. Hence, we are forced to consider the case “+” for  $n$  even and “−” for “ $n$ ” odd.

We will focus on these cases. The Maurer-Cartan element provides us with a differential  $d$ . We define the dg Lie algebras

$$\mathbf{fGC}_n = (\mathbf{fGC}_{n,n}^{(\pm)^n}, d).$$

**Remark.** Concretely up to prefactors, the differential applied to some graph  $\Gamma$  has the form

$$d\Gamma = \sum_{v \in V(\Gamma)} \frac{1}{2} (\text{splitting of } v) - (\text{adding an edge at } v).$$

Here the sum runs over all vertices of  $\Gamma$ . The “(splitting of  $v$ )” means the vertex  $v$  is replaced by a pair of vertices connected by an edge, and the incoming edges at  $v$  are distributed arbitrarily among the two newly created vertices. The term “(adding an edge at  $v$ )” stands for a graph obtained by adding a new vertex and connecting it to  $v$ . If there are any incoming edges at  $v$ , the second term cancels

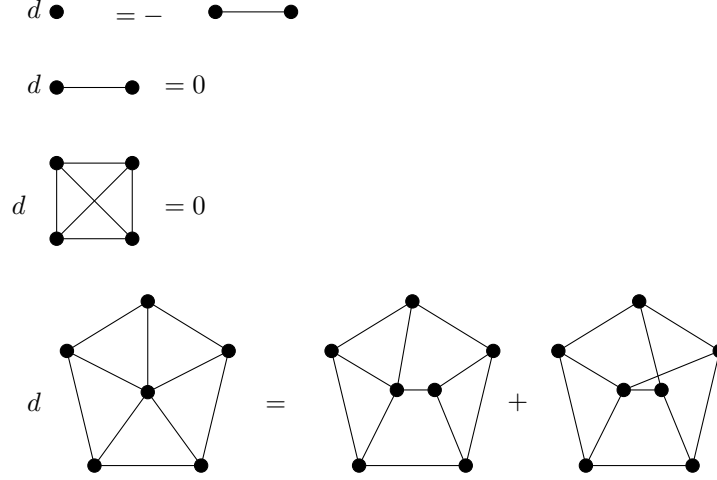


FIGURE 3. Several example computations of the graph differential. Note that in drawing these pictures we are cheating a bit since we did not say what element of the graph complex the picture of a graph stands for. However, this is “merely” a matter of prefactors. The prefactors shown here can be recovered up to sign by interpreting a graph with unnumbered vertices as the sum over all numberings of vertices, divided by the order of the automorphism group.

those graphs from the first term, in which all incoming edges at  $v$  had been connected to one of the newly added vertices. Several examples can be found in Figure 3.

Every graph can be split into a product of its connected components. Hence we can write

$$\mathrm{fGC}_n = S^+(\mathrm{fGC}_{n,\mathrm{conn}}[-n])[n]$$

where  $\mathrm{fGC}_{n,\mathrm{conn}}$  is the subcomplex of  $\mathrm{fGC}_n$  generated by the connected graphs. Note that so far our graphs may contain vertices of any valence, and possibly tadpoles (i.e., edges  $(j, j)$ ) and multiple edges. More precisely, by symmetry reasons, there can be tadpoles only for  $n$  even, and multiple edges only for  $n$  odd. We define  $\mathrm{GC}_n$  to be the subspace of  $\mathrm{fGC}_{n,\mathrm{conn}}$  spanned by those graphs with all vertices of valence at least 3, with no tadpoles, but we *allow* multiple edges. The following proposition has (mostly) been proven by M. Kontsevich [11, 12].

**Proposition 3.**  *$\mathrm{GC}_n$  is a sub-dg Lie algebra. The cohomology satisfies*

$$H(\mathrm{fGC}_{n,\mathrm{conn}}) = H(\mathrm{GC}_n) \oplus \bigoplus_{\substack{j \geq 1 \\ j \equiv 2n+1 \pmod{4}}} \mathbb{R}[n-j].$$

Here, the class corresponding to  $\mathbb{R}[n-j]$  is represented by a loop with  $j$  edges, as in Figure 5.

*Sketch of proof.* Part of the proof is copied from [11, 12]. First note that the differential does not produce any vertices of valence 1, nor can it produce tadpoles or multiple edges if there were none in the graph before. Assume a graph  $\Gamma$  with all vertices at least trivalent is given. Then the differential  $d\Gamma$  contains graphs with one bivalent vertex, one for every edge. However, each such graph comes twice with opposing signs. Hence we can conclude that  $d\mathrm{GC}_n \subset \mathrm{GC}_n$ . This shows that

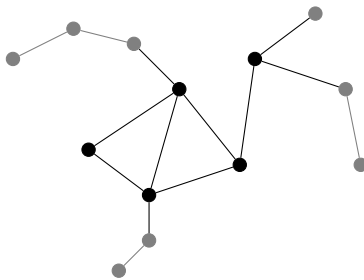
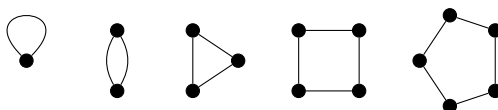


FIGURE 4. The “antennas” in the graph are drawn in gray.

FIGURE 5. Some graphs from the subcomplex of “loops”, as occurring in the proof of Proposition 3. Note: Some of these graphs (depending on  $n$ ) are zero by symmetry.

$\text{GC}_n$  is indeed a subcomplex. Similarly, Next Let  $\text{GC}_n^1$  be the space spanned by graphs having at least one vertex of valence 1, and let  $\text{GC}_n^2$  be the space spanned by graphs having no vertex of valence one, but at least one vertex of valence two. Such vertices cannot be killed by the differential and hence we have decomposition (of complexes)

$$\text{fGC}_{n,\text{conn}} \cong \text{GC}_n^{\geq 3} \oplus \text{GC}_n^2 \oplus \text{GC}_n^1$$

where  $\text{GC}_n^{\geq 3}$  is spanned by graphs containing only at least trivalent vertices.

We claim that  $\text{GC}_n^1$  is acyclic. Indeed for the subcomplex of graphs not containing a trivalent vertex (i.e., “linear graphs”), this is easily shown. Assume next that a graph, say  $\Gamma$ , has at least one trivalent vertex. We call an “antenna” a maximal connected subgraph consisting of one- and two-valent vertices in  $\Gamma$ . Then full graph  $\Gamma$  can be seen as some “core graph” (the complement of the union of all antennas) with antennas of various length attached. See Figure 4 for a graphical illustration of those terms. One can set up a spectral sequence such that the first differential is the one increasing the sum of the lengths of the antennas. It is easily seen that this complex is acyclic and hence the claim is shown.<sup>2</sup>

Next we claim that the cohomology of  $\text{GC}_n^2$  is

$$H(\text{GC}_n^2) = \bigoplus_{\substack{j \geq 1 \\ j \equiv 2n+1 \pmod{4}}} \mathbb{R}[n-j].$$

Indeed one checks that this is the cohomology of the subcomplex of “loops” (see Figure 5). Hence the claim reduces to showing that the subcomplex of  $\text{GC}_n^2$  spanned by graphs having at least one trivalent vertex is acyclic. Any such graph can be written as a “core” with only trivalent vertices, and the edges labeled by natural numbers. An edge labelled by  $k$  represents a “string” of  $k-1$  bivalent vertices. One can set up a spectral sequence such that the first differential increases one of the labels by one. More precisely, the differential will increase only any even label and map oddly labelled edges to zero. It is easily checked that the resulting complex is

<sup>2</sup>Worries about the convergence of this spectral sequence are addressed in Appendix E.

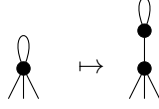


FIGURE 6. Picture of the part of the differential on  $\mathrm{GC}_n^{\geq 3}$  producing a valence three tadpole vertex, as occurring in the proof of Proposition 3.

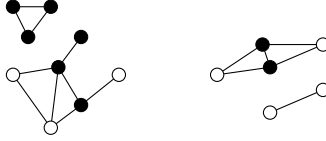


FIGURE 7. A typical graph from  $\mathrm{fGraphs}_n$  (left) and one from the sub-operad  $\mathrm{Graphs}_n$  (right).

indeed acyclic. Here some care has to be taken in case some labelled edge forms a tadpole, but the argument still works.

Finally consider  $\mathrm{GC}_n^{\geq 3}$ . We still need to show that we can omit the tadpoles. One can set up a spectral sequence coming from the filtration according to the number of tadpole vertices of valence three. The first differential hence produces such a vertex, see Figure 6. Similar arguments as for  $\mathrm{GC}_n^1$  then show that the cohomology is given by the tadpole-free graphs. This proves the proposition.  $\square$

The complexes  $\mathrm{GC}_n$  are Kontsevich’s (cohomological) graph complexes, see [11, 12, 13]. We note that the  $\mathrm{GC}_n$  for various even  $n$  are isomorphic, up to some shifting of degrees, as are those for various odd  $n$ . However, there is no map to our knowledge relating the cohomology of the even  $n$  to that of the odd  $n$  complexes.

**Remark.** As probably<sup>3</sup> noted by M. Kontsevich [13], and proven by Conant et al. [4], the subcomplex of  $\mathrm{GC}_n$  given by 1-vertex irreducible graphs is quasi-isomorphic to  $\mathrm{GC}_n$ . We give a short sketch of a different proof in Appendix D.

Often in the literature, tadpoles are excluded from the graph complex from the start. We will have use for the tadpole-free version of the graph complex later and introduce the following notation: For  $n$  even,  $\mathrm{Gra}_n$  is the suboperad of  $\mathrm{Gra}_n^-$  spanned by tadpole-free graphs, while  $\mathrm{Gra}_n^\circ = \mathrm{Gra}_n^-$  is the full operad. For  $n$  odd,  $\mathrm{Gra}_n = \mathrm{Gra}_n^\circ$  is simply  $\mathrm{Gra}_n^+$ . (There are no tadpoles by symmetry reasons anyways.) In general, whenever we draw a symbol “ $\circ$ ” beneath some graph space, it means that tadpoles are allowed.

**2.1. The (twisted) operads  $\mathrm{Graphs}_n$ .** The general theory of twisting (see Appendix G) produces from the operads  $\mathrm{Gra}_n$  new operads  $\mathrm{fGraphs}_n$ . Generators of  $\mathrm{fGraphs}_n(N)$  (as vector space) can be depicted by graphs with two kinds of vertices: (i) “external” vertices, which are numbered  $1, \dots, N$  and (ii) “internal” vertices, which are “indistinguishable”. In pictures, we draw external vertices white and internal vertices black. For an example, see Figure 7. The operadic composition is obtained by inserting at external vertices. Again by the general theory there is a dg action of the dg Lie algebra  $\mathrm{fGC}_n$  on the operad  $\mathrm{fGraphs}_n$ . Loosely speaking, the

<sup>3</sup>He announced a proof that probably contains this fact.

action is the same as the adjoint action on  $\mathbf{fGC}_n$ , except that some vertices happen to be colored white.

Similarly to the case of graph complexes, we want to simplify the operad  $\mathbf{fGraphs}_n$  a bit. Let  $\mathbf{Graphs}_n^\circ$  be the suboperad spanned by graphs with all internal vertices at least trivalent and with no connected component consisting entirely of internal vertices. Let  $\mathbf{Graphs}_n$  be the same thing, except that graphs with tadpoles are also excluded.

**Lemma 1.**  $\mathbf{Graphs}_n^\circ$  and  $\mathbf{Graphs}_n$  are dg suboperads.

*Proof.* One notes that neither of the forbidden things can be created by insertions or the differential.  $\square$

Note that there is a splitting

$$\mathbf{fGraphs}_n(N) = \mathbf{fGraphs}_{n,c}(N) \otimes (\mathbb{R} \oplus \mathbf{fGC}_n[-n])$$

where  $\mathbf{fGraphs}_{n,c}$  is the suboperad consisting of graphs with no connected components consisting entirely of internal vertices. Copying the proof of Proposition 3, one can show the following.

**Proposition 4.** The inclusions  $\mathbf{Graphs}_n \subset \mathbf{Graphs}_n^\circ \subset \mathbf{fGraphs}_{n,c}$  are quasi-isomorphisms.

The cohomology of  $\mathbf{Graphs}_n$  has been computed by M. Kontsevich [14] and also P. Lambrechts and I. Volic [16].

**Proposition 5** (Kontsevich, Lambrechts and Volic). *The cohomology of  $\mathbf{Graphs}_n$  is the operad governing  $n$ -algebras, i.e.,*

$$H(\mathbf{Graphs}_n) = e_n.$$

**Remark.** We note in particular that the cohomology of  $\mathbf{Graphs}_2^\circ$  is the Gerstenhaber operad  $e_2$  and not the BV operad. However, if one quotients out from  $\mathbf{Graphs}_2^\circ$  the space spanned by graphs with a tadpole at an internal vertex, the cohomology is the BV operad.

Following [18], one notes that each graph in  $\mathbf{Graphs}_n$  decomposes into a (co)product of internally connected components. Here “internally connected” means connected after deleting all external vertices. One can hence write

$$\mathbf{Graphs}_n = S(\mathbf{ICG}_n[1])$$

where  $\mathbf{ICG}_n$  is spanned by internally connected graphs, shifted in degree by 1. One checks that the differential is compatible with the coproduct and hence the  $\mathbf{ICG}_n$  form (operads of)  $L_\infty$  algebras.

The following proposition was shown in [18] for  $n = 2$ .

**Proposition 6.** *The cohomologies of the  $\mathbf{ICG}_n$  are the (operads of) graded Lie algebras  $\mathfrak{t}^{(n)}$ , where  $\mathfrak{t}^{(n)}(N)$  is generated by symbols  $t_{ij} = (-1)^{n-1}t_{ji}$ ,  $1 \leq i \neq j \leq N$ , of degree  $2 - n$ , with relations  $[t_{ij}, t_{ik} + t_{jk}] = 0$  for  $\#\{i, j, k\} = 3$  and  $[t_{ij}, t_{kl}] = 0$  for  $\#\{i, j, k, l\} = 4$ .*

*Proof.* Copy the proof of the Appendix of [18] and change the gradings.  $\square$

**Remark.** Here it is crucial that we allow multiple edges for  $n$  odd. Otherwise it is not true that  $\mathbf{Graphs}_n = S(\mathbf{ICG}_n[1])$ .

### 3. THE MAP BETWEEN $\mathrm{GC}_n$ AND $\mathrm{Def}(e_n)$

**3.1. Deformations of operads and operad maps.** Let  $\mathcal{C}$  be a cooperad and  $\mathcal{P}$  be an operad. Let  $\Omega(\mathcal{C})$  be the cobar construction of  $\mathcal{C}$ , which is a quasi-free operad. Let  $f: \Omega(\mathcal{C}) \rightarrow \mathcal{P}$  be a morphism of operads. Then following [17] one can define a *deformation complex* of  $f$ :

$$\mathrm{Def}(\Omega(\mathcal{C}) \xrightarrow{f} \mathcal{P}).$$

It is a dg Lie algebra. Its first cohomology governs the infinitesimal deformations of  $f$ . Its Maurer-Cartan elements correspond to operad morphisms  $\Omega(\mathcal{C}) \rightarrow \mathcal{P}$ .

Next consider the special case  $\mathcal{P} = \Omega(\mathcal{C})$ . One can define a complex:

$$\mathrm{Def}(\Omega(\mathcal{C})) := \mathrm{Def}(\Omega(\mathcal{C}) \xrightarrow{id} \Omega(\mathcal{C}))[1].$$

Elements  $f$  of this complex can be considered as infinitesimal deformations of the identity morphism  $id + \epsilon f$ . Two such morphisms can be composed. This defines another Lie bracket on  $\mathrm{Def}(\Omega(\mathcal{C}))$ :

$$(id + \epsilon f) \circ (id + \epsilon g) - (id + \epsilon g) \circ (id + \epsilon f) =: id + \epsilon^2 [f, g] \quad \text{modulo } \epsilon^3.$$

This bracket makes  $\mathrm{Def}(\Omega(\mathcal{C}))$  into a dg Lie algebra. For more details see [21]. Note that for constructing this second bracket the quasi-freeness of  $\Omega(\mathcal{C})$  is not needed. In the present paper we are mainly interested in the case  $\mathcal{C} = e_n^\vee$  and hence  $\Omega(\mathcal{C}) =: hoe_n$  is the minimal cofibrant resolution of the operad of  $n$ -algebras,  $n = 2, 3, \dots$ . We define

$$\mathrm{Def}(e_n) := \mathrm{Def}(\Omega(\mathcal{C})).$$

This complex governs the deformations up to homotopy of the operad  $e_n$ .

**3.2. Reduction to the connected part.** There are canonical quasi-isomorphisms  $hoe_n \rightarrow e_n \rightarrow \mathrm{Graphs}_n$ , and hence also quasi-isomorphisms

$$\mathrm{Def}(e_n)[-1] \rightarrow \mathrm{Def}(hoe_n \rightarrow e_n) \rightarrow \mathrm{Def}(hoe_n \rightarrow \mathrm{Graphs}_n).$$

Elements of the complex on the right can be written as linear combinations of graphs, with certain symmetry properties. See Appendix B for a more detailed description. Any such graph splits into a union of its connected components, and the differential acts on each component separately and preserves connectedness. Let  $\mathrm{Def}(hoe_n \rightarrow \mathrm{Graphs}_n)_{\mathrm{conn}} \subset \mathrm{Def}(hoe_n \rightarrow \mathrm{Graphs}_n)$  be the subspace spanned by the connected graphs. Then we can write

$$\mathrm{Def}(hoe_n \rightarrow \mathrm{Graphs}_n) \cong S^+(\mathrm{Def}(hoe_n \rightarrow \mathrm{Graphs}_n)_{\mathrm{conn}}[-n])[n].$$

By the canonical action of  $\mathrm{GC}_n$  on  $\mathrm{Graphs}_n$ , there is a map

$$\Phi: \mathrm{GC}_n \rightarrow \mathrm{Def}(hoe_n \rightarrow \mathrm{Graphs}_n)_{\mathrm{conn}}[1].$$

Theorem 2 states that this map is a quasi-isomorphism, up to “uninteresting” terms. For the following proof it will be slightly more economic to allow tadpoles in our graphs, i.e., to work with  $\mathrm{Graphs}_n^\circ$  instead of  $\mathrm{Graphs}_n$ . Since the inclusion  $\mathrm{Graphs}_n \rightarrow \mathrm{Graphs}_n^\circ$  is a quasi-isomorphism, this is not a severe change.

**3.3. Proof of Theorem 2.** There are natural maps  $hoLie_n \rightarrow hoe_n$  and  $e_n \rightarrow \mathrm{Graphs}_n^\circ \rightarrow \mathrm{Gra}_n^\circ$ . They allow one to write a sequence of maps

$$(1) \quad 0 \rightarrow \mathrm{Def}(hoe_n \rightarrow e_n) \rightarrow \mathrm{Def}(hoe_n \rightarrow \mathrm{Gra}_n^\circ) \rightarrow \mathrm{Def}(hoLie_n \rightarrow \mathrm{Gra}_n^\circ) \rightarrow 0.$$

All these spaces can be written as (graded) symmetric powers of their connected parts. In this section, we will only care about the latter.

$$(2) \quad 0 \rightarrow \mathrm{Def}(hoe_n \rightarrow e_n)_{\mathrm{conn}} \rightarrow \mathrm{Def}(hoe_n \rightarrow \mathrm{Gra}_n^\circ)_{\mathrm{conn}} \rightarrow \mathrm{Def}(hoLie_n \rightarrow \mathrm{Gra}_n^\circ)_{\mathrm{conn}} \rightarrow 0.$$



The above sequence is exact on the left and right, but not in the middle. The cohomology of the middle term is calculated in Proposition 21 to be  $\mathbb{R}[-1]$ . Furthermore this  $\mathbb{R}[-1]$  is mapped to zero in  $H^1(\mathrm{Def}(hoe_n \rightarrow \mathrm{Gra}_n^\odot)_{conn})$ . Assume now that the sequence (2) was exact. Then from the corresponding long exact sequence in homology one could conclude that

$$H^{\bullet+1}(\mathrm{Def}(hoe_n \rightarrow e_n)_{conn}) = H^\bullet(\mathrm{Def}(hoe_n \rightarrow \mathrm{Gra}_n^\odot)_{conn}) \oplus \mathbb{R}[-1].$$

Hence, using that

$$H(\mathrm{Def}(hoe_n \rightarrow \mathrm{Gra}_n^\odot)_{conn}) \cong H(\mathrm{GC}_n) \bigoplus_{\substack{j \geq 1 \\ j \equiv 2n+1 \pmod{4}}} \mathbb{R}[n-j]$$

by Proposition 3, Theorem 2 would be proven. The problem is now that (2) is *not* exact in the middle. However, one can cure that defect:

**Proposition 7.** *There is a long exact sequence in homology*

$$\begin{aligned} \cdots \rightarrow H^k(\mathrm{Def}(hoe_n \rightarrow \mathrm{Gra}_n)_{conn}) &\rightarrow H^k(\mathrm{Def}(hoe_n \rightarrow \mathrm{Gra}_n^\odot)_{conn}) \rightarrow \\ &\xrightarrow{\phi} H^{k+1}(\mathrm{Der}(hoe_n \rightarrow e_n)_{conn}) \rightarrow H^{k+1}(\mathrm{Def}(hoe_n \rightarrow \mathrm{Gra}_n)_{conn}) \rightarrow \cdots \end{aligned}$$

On graphs  $\Gamma \in \mathrm{GC}_n \subset \mathrm{Def}(hoe_n \rightarrow \mathrm{Gra}_n)$ , the connecting morphism  $\phi$  agrees with the map  $\Phi$  defined above.<sup>4</sup>

For the proof, one can apply the following Lemma, which the author learned from D. Kazhdan.

**Lemma 2.** *Let*

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

*be a sequence of complexes such that the composition of consecutive arrows is 0. Assume that the total cohomology of the double complex  $D = (A[-1] \oplus B \oplus C[1])$ , vanishes. Then there is a long exact sequence in cohomology*

$$\cdots \rightarrow H^k(B) \rightarrow H^k(C) \rightarrow H^{k+1}(A) \rightarrow H^{k+1}(B) \rightarrow \cdots$$

*Proof (Sketch).* Compute the cohomology of the double complex  $D$  using the associated spectral sequence. The  $E^1$ -term is

$$0 \rightarrow H(A) \rightarrow H(B) \rightarrow H(C) \rightarrow 0.$$

The spectral sequence must collapse at the  $E^3$  by degree reasons, and hence  $E^3 = 0$  by the assumption in the Lemma. (And the fact that spectral sequences of bounded double complexes converge to the true cohomology.) Concretely, the nontrivial parts of the complex  $E^2$  are

$$\begin{aligned} 0 \rightarrow \ker(H(B) \rightarrow H(C)) / \mathrm{Im}(H(A) \rightarrow H(B)) &\rightarrow 0 \\ 0 \rightarrow \ker(H(A) \rightarrow H(B)) \rightarrow \mathrm{coker}(H(B) \rightarrow H(C)) &\rightarrow 0. \end{aligned}$$

From the vanishing of the cohomology of  $E^3$  the exactness of the long sequence then follows.  $\square$

Let us now turn to the proof of Proposition 7. We want to apply the above Lemma to  $A = \mathrm{Def}(hoe_n \rightarrow e_n)_{conn}$ ,  $B = \mathrm{Def}(hoe_n \rightarrow \mathrm{Gra}_n^\odot)_{conn}$  and  $C = \mathrm{Def}(hoe_n \rightarrow \mathrm{Gra}_n)_{conn}$ . In order to do so, we need to show that the cohomology of the double complex  $D$  (as in the Lemma) vanishes. Take the spectral sequence whose first term is the (horizontal) cohomology of

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0.$$

<sup>4</sup>Here the identification  $H(\mathrm{Def}(hoe_n \rightarrow e_n)) \cong H(\mathrm{Def}(hoe_n \rightarrow \mathrm{Graphs}_n))$  is used.

Since in our case that sequence is already exact on the left and right, this cohomology is

$$E^1 := \ker(B \rightarrow C)/\text{Im}(A \rightarrow B) = \text{Def}'(\text{hoe}_n \rightarrow (\text{Gra}_n^\odot/e_n))_{\text{conn}}$$

where the  $'$  indicates that one restricts to those derivations with vanishing  $\text{hoLie}_n$ -part. By degree reasons the spectral sequence abuts at  $E^2$ , hence we have to show that  $E^2 = H(E^1) = 0$ .

**Lemma 3.** *Indeed  $H(E^1) = 0$ .*

*Proof.* This computation is the technical heart of the proof. We will use here the notation, in particular the graphical representation of  $\text{Def}(\text{hoe}_n \rightarrow \text{Gra}_n^\odot)$ , from Appendix B. Furthermore, let  $\Xi \subset \text{Def}(\text{hoe}_n \rightarrow e_n)$  the subcomplex of Appendix A. It was shown in Proposition 20 that the inclusion is a quasi-isomorphism. Define  $\Xi_{\text{conn}} \subset \Xi$  to be the connected part. We will show that the inclusion  $\Xi_{\text{conn}} \rightarrow \text{Def}'(\text{hoe}_n \rightarrow \text{Gra}_n^\odot)_{\text{conn}}$  is also a quasi-isomorphism. In fact, there is a spectral sequence whose second convergent is  $\Xi_{\text{conn}}$ . This spectral sequence comes from the filtration on  $\text{Def}'(\text{hoe}_n \rightarrow \text{Gra}_n^\odot)_{\text{conn}}$  by the number  $k_1$  of clusters of length 1. The differential can increase  $k_1$  by at most one. More concretely, the part of the differential, say  $d_+$ , that increases  $k_1$  by one is the part that splits off a length one cluster from any vertex. Let us introduce new terminology. Let us call vertices in clusters of length 1 “internal” and all others external. Then we see that our complex is in fact isomorphic to some subcomplex of the version  $\text{Graphs}'$  of the graph-complex without the trivalence condition on internal vertices. Hence the cohomology is given by closed graphs without internal vertices. In other words, graphs with all clusters of length  $\geq 2$ , and which actually lie in  $\text{Def}(\text{hoe}_n \rightarrow e_n)_{\text{conn}} \subset \text{Def}(\text{hoe}_n \rightarrow \text{Gra}_n^\odot)_{\text{conn}}$ . But that space is the space we call  $\Xi_{\text{conn}}$ .  $\square$

#### 4. THE GROTHENDIECK-TEICHMÜLLER LIE ALGEBRA

**4.1. The standard definition.** Let  $\mathbb{F}_2 = k\langle\langle x, y \rangle\rangle$  be the completed free algebra in generators  $x, y$ . There is a coproduct  $\Delta$  on  $\mathbb{F}_2$  by declaring  $x, y$  to be primitive, i.e.,  $\Delta x = x \otimes 1 + 1 \otimes x$ ,  $\Delta y = y \otimes 1 + 1 \otimes y$ . We call an element  $\Phi \in \mathbb{F}_2$  *group-like* if  $\Delta\Phi = \Phi \otimes \Phi$ . Equivalently,  $\phi$  is grouplike if  $\Phi = \exp(\phi)$  with  $\phi \in \hat{\mathbb{F}}_{\text{Lie}}(x, y) \subset \mathbb{F}_2$ .

Let  $\mathfrak{t}_n$  ( $n = 2, 3, \dots$ ) be the Drinfeld-Kohno Lie algebra. It is generated by symbols  $t_{ij} = t_{ji}$ ,  $1 \leq i, j \leq n$ ,  $i \neq j$ , with relations  $[t_{ij}, t_{ik} + t_{kj}] = 0$  for  $\#\{i, j, k\} = 3$  and  $[t_{ij}, t_{kl}] = 0$  for  $\#\{i, j, k, l\} = 4$ . Consider the following set of equations for elements  $\Phi \in \mathbb{F}_2$ , depending on some yet unspecified parameter  $\mu \in k$ .

(3)

$$\Phi(t_{12}, t_{23} + t_{24})\Phi(t_{13} + t_{23}, t_{34}) = \Phi(t_{23}, t_{34})\Phi(t_{12} + t_{13}, t_{24} + t_{34})\Phi(t_{12}, t_{23})$$

(4)

$$e^{\mu(t_{13}+t_{23})/2} = \Phi(t_{13}, t_{12})e^{\mu t_{13}/2}\Phi(t_{13}, t_{23})^{-1}e^{\mu t_{23}/2}\Phi(t_{12}, t_{23})$$

(5)

$$\Phi(x, y) = \Phi(y, x)^{-1}$$

**Definition 1.** *The group-like solutions  $\Phi \in \mathbb{F}_2$  of (3), (4), (5) are called Drinfeld associators for  $\mu \neq 0$  and elements of the Grothendieck-Teichmüller group GRT for  $\mu = 0$ .*

H. Furusho has shown the following remarkable Theorem.

**Theorem 3** (Furusho [9]). *Any group-like solution  $\Phi \in \mathbb{F}_2$  of (3) automatically satisfies (4) and (5) for  $\mu = \pm\sqrt{24c_2(\Phi)}$ , where  $c_2(\Phi)$  is the coefficient of  $xy$  in  $\Phi$ .*

We reprove the  $\mu = 0$  case of this result in Appendix C.

**Remark.** It also follows from (3) that  $\Phi$  contains no terms linear in  $x, y$ , i.e.,

$$\Phi(x, y) = 1 + c_2(\Phi)(xy - yx) + (\text{higher orders}).$$

As the name suggests, there is a group structure on the Grothendieck-Teichmüller group  $\mathbf{GRT}$  as follows.

$$(\Phi_1 \cdot \Phi_2)(x, y) = \Phi_1(x, y) \Phi_2(\Phi_1(x, y)^{-1} x \Phi_1(x, y), y).$$

We will actually be mostly interested in its Lie algebra, the Grothendieck-Teichmüller Lie algebra  $\mathbf{grt}$ . It is given by Lie series  $\phi \in \hat{\mathbb{F}}_{Lie}(x, y)$  such that the following hold.

(6)

$$\phi(t_{12}, t_{23} + t_{24}) + \phi(t_{13} + t_{23}, t_{34}) = \phi(t_{23}, t_{34}) + \phi(t_{12} + t_{13}, t_{24} + t_{34}) + \phi(t_{12}, t_{23})$$

(7)

$$\phi(x, y) + \phi(y, -x - y) + \phi(-x - y, x) = 0$$

(8)

$$\phi(x, y) + \phi(y, x) = 0$$

Again, by Furusho's result, it suffices to require (6) and that  $\phi$  contains no quadratic term (i.e., no  $[x, y]$ ).

**4.2. Definition as cohomology of  $\mathbf{t}$ .** Let  $\hat{\mathbf{t}}_n$  be the completion of  $\mathbf{t}_n$ . The spaces  $\hat{\mathbf{t}}_n$  in fact form an operad  $\hat{\mathbf{t}}$  of Lie algebras. Hence the Chevalley complexes  $C(\hat{\mathbf{t}})$  form an operad of coalgebras, and in particular an operad of vector spaces. There is a map  $Com \rightarrow C(\hat{\mathbf{t}})$  of operads of vector spaces, sending the generator of  $Com$  to  $1 \in \mathbf{t}_2$ . Hence one also obtains a map  $Com_\infty \rightarrow C(\hat{\mathbf{t}})$ , and can consider the deformation complex  $\text{Def}(Com_\infty \rightarrow C(\hat{\mathbf{t}}))$ . This complex inherits a grading from the grading on  $C(\hat{\mathbf{t}})$ . By abuse of notation, we denote the degree 1 part

$$\text{Def}(Com_\infty \rightarrow \hat{\mathbf{t}}[1]).$$

The following Proposition was shown by D. Tamarkin to the knowledge of the author.

**Proposition 8.**  $H^1(\text{Def}(Com_\infty \rightarrow \hat{\mathbf{t}}[1])) \cong \mathbf{grt}$ .

*Proof.* By degree reasons,  $H^1(\text{Def}(Com_\infty \rightarrow \hat{\mathbf{t}}[1]))$  is the space of closed elements in

$$\text{Hom}_{S_3}(Com^\vee(3), \hat{\mathbf{t}}_3[1])$$

modulo exact elements. From the fact that  $\mathbf{t}_2$  is one-dimensional, one can show that there are no exact elements. Hence one needs to compute closed elements in  $\hat{\mathbf{t}}_3$  with the correct symmetry properties. However, one checks that the required symmetries are exactly equations (7) and (8), and closedness is exactly (6).  $\square$

In [18] we introduced  $L_\infty$  algebras  $\mathbf{ICG}(n)$ , forming an operad of  $L_\infty$ -algebras, such that  $H(\mathbf{ICG}) = \hat{\mathbf{t}}$ . We will use here the (quasi-isomorphic) version with tadpoles  $\mathbf{ICG}^\circ$  for technical reasons.<sup>5</sup> There is again a map  $Com \rightarrow \mathbf{ICG}^\circ$ . Using a spectral sequence argument one can derive from the previous Proposition the following.

**Proposition 9.**  $H^1(\text{Def}(Com_\infty \rightarrow \mathbf{ICG}^\circ[1])) \cong \mathbf{grt}$ .

An element of  $\text{Def}(Com_\infty \rightarrow \mathbf{ICG}^\circ)$  is given by a collection of maps in

$$\text{Hom}_{S_n}(Com^\vee(n), \mathbf{ICG}^\circ(n)[1]).$$

Such maps can be depicted as internally connected graphs in  $\mathbf{Graphs}_2(n)$ , satisfying certain symmetry conditions under interchange of their external vertices. More precisely, they are required to vanish on shuffles: For any  $k, l \geq 1$ ,  $k + l = n$ , one requires

$$\sum_{\sigma \in \text{ush}(k, l)} \text{sgn}(\sigma) \sigma \cdot \Gamma = 0.$$

<sup>5</sup>One can do everything without the tadpoles, but it is less elegant.

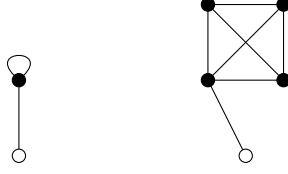


FIGURE 8. Some graphs in the complex  $C \subset \text{Def}(Com_\infty \rightarrow \text{ICG}^\circ[1])$ .

Here the sum is over  $(k, l)$ -unshuffle permutations, and a permutation acts on graphs  $\Gamma$  by interchange of the labels on the external vertices. One can define a subcomplex  $C \subset \text{Def}(Com_\infty \rightarrow \text{ICG}^\circ[1])$  consisting of graphs with one external vertex and exactly one edge connecting to them. Some example graphs in  $C$  are shown in Figure 8.

**Proposition 10.** *The inclusion  $C \hookrightarrow \text{Def}(Com_\infty \rightarrow \text{ICG}^\circ[1])$  is a quasi-isomorphism. Hence  $H^1(C) \cong \mathfrak{grt}$ .*

*Proof.* It follows along the lines of the more general proof of Proposition 22. There one also sees what has to be changed to circumvent using tadpoles. (To  $C$  one would have to add the graph with two external vertices and one edge.)  $\square$

Furthermore, note that (see [18])  $\text{Graphs}^\circ(n) = C(\text{ICG}(n))$  is the (completed) bar construction. Hence  $\text{Def}(Com_\infty \rightarrow \text{ICG}^\circ[1]) \subset \text{Hom}(Com_\infty \rightarrow \text{Graphs}^\circ)$ .

**Proposition 11.** *The inclusion  $C \hookrightarrow \text{Def}(Com_\infty \rightarrow \text{Graphs}^\circ)$  is a quasi-isomorphism. Hence*

$$H^1(\text{Def}(Com_\infty \rightarrow \text{Graphs}^\circ)) \cong \mathfrak{grt}.$$

*Proof.* It is a copy of the proof of Proposition 10.  $\square$

#### 4.3. Tamarkin's $\mathfrak{grt}$ -action (up to homotopy) on $G_\infty$ and on $T_{\text{poly}}$ .

4.3.1. *On  $G_\infty$ .* As noted by D. Tamarkin [21], there is an action of  $\mathfrak{grt}$  on the homotopy Gerstenhaber operad  $G_\infty = \text{hoe}_2$ . Let  $PaP_n$  be the category with objects the parenthesized permutations of symbols  $1, \dots, n$  (for example  $(13)(5(24))$ ) and with exactly one morphism between any pair of objects. Let  $PaCD_n := U\mathfrak{t} \times PaP$  be the product of categories, where the completed universal enveloping algebra  $U\mathfrak{t}_n$  of  $\mathfrak{t}_n$  is considered as a category with one object.<sup>6</sup> The  $PaCD_n$  assemble to form an operad of categories enriched over (complete) Hopf algebras, called  $PaCD$ . The Grothendieck-Teichmüller group  $\text{GRT}$  can be identified with its automorphism group. In particular, one has an action of  $\mathfrak{grt}$  on  $PaCD$ . See [2] for details.

Furthermore, there is a chain of quasi-isomorphisms of operads

$$CNPaCD \rightarrow BU\mathfrak{t} \leftarrow e_2 \leftarrow G_\infty.$$

Here “ $\mathcal{N}$ ” denotes the nerve and “ $C$ ” the chains functor. The left arrow is obtained from the map  $PaCD \rightarrow U\mathfrak{t}$  sending  $PaP$  to the category with one object and one morphism. Since  $G_\infty$  is cofibrant, the action of  $\mathfrak{grt}$  on  $CNPaCD$  can be transferred to an action (up to homotopy) on  $G_\infty$ . One hence obtains an  $L_\infty$  morphism

$$\mathfrak{grt} \rightarrow \text{Def}(G_\infty).$$

D. Tamarkin proved the following theorem.

**Theorem 4** (Tamarkin [21]). *The above morphism  $\mathfrak{grt} \rightarrow \text{Der}(G_\infty)$  is homotopy injective.*

<sup>6</sup> $PaCD$  stands for “parenthesized chord diagrams”, see [2].

4.3.2. *On  $T_{\text{poly}}$ .* Let next  $T_{\text{poly}} = \Gamma(\mathbb{R}^d; \wedge T\mathbb{R}^d)$  be the space of polyvector fields on  $\mathbb{R}^d$ . It is naturally a Gerstenhaber algebra, hence we have maps  $G_\infty \rightarrow e_2 \rightarrow \text{End}(T_{\text{poly}})$ , where  $\text{End}(T_{\text{poly}})$  is the endomorphism operad. From the  $\mathbf{grt}$ -action on the left one hence obtains a map

$$A: \mathbf{grt} \rightarrow \text{Def}(G_\infty \rightarrow \text{End}(T_{\text{poly}}))[1].$$

Since the  $G_\infty$ -structure on  $T_{\text{poly}}$  is not ( $GL_d$ -equivariantly) deformable, the image of this map will be exact. So, for any  $\phi \in \mathbf{grt}$  there will be some  $h \in \text{Der}(G_\infty \rightarrow \text{End}(T_{\text{poly}}))$  such that  $A(\phi) = dh$ .

**Remark.** The homotopy  $h$  encodes an infinitesimal  $G_\infty$  map between the two  $G_\infty$  structures on  $T_{\text{poly}}$  related by  $\phi$ .

By degree reasons,  $\phi$  acts trivially on the  $L_\infty$  part of the  $G_\infty$ -structure on  $T_{\text{poly}}$ , i.e., it is mapped to zero under

$$\text{Def}(G_\infty \rightarrow \text{End}(T_{\text{poly}})) \rightarrow \text{Def}(L_\infty \rightarrow \text{End}(T_{\text{poly}})).$$

We can hence conclude that  $dh' = 0$  where  $h' \in \text{Def}(L_\infty \rightarrow \text{End}(T_{\text{poly}}))$  is the image of  $h$  under the above map. In other words,  $h'$  encodes an  $L_\infty$  derivation on  $T_{\text{poly}}$ . One can check that  $h'$  is determined uniquely up to homotopy by  $\phi$ , again using rigidity of  $T_{\text{poly}}$ . Hence one has an action (up to homotopy) of  $\mathbf{grt}$  on  $T_{\text{poly}}$  by  $L_\infty$ -derivations. This action was discovered by D. Tamarkin (to my knowledge).

4.4. **D. Tamarkin's action on  $G_\infty$  revisited.** There are quasi-isomorphisms of vector spaces

$$\text{Def}(G_\infty)[-1] \rightarrow \text{Def}(G_\infty \rightarrow G_\infty) \rightarrow \text{Def}(G_\infty \rightarrow e_2) \leftarrow \text{Def}(G_\infty \rightarrow \text{Graphs}_2).$$

Let  $C$  be the complex from section 4.2 computing the simplicial cohomology of  $\mathbf{t}$ . In particular,  $H^1(C) \cong \mathbf{grt}$ . Elements of  $C$  are given by graphs in  $\text{ICG}^\circ$  with one external vertex, which has valence 1. One has the natural inclusion<sup>7</sup>

$$C \hookrightarrow \text{Def}(G_\infty \rightarrow \text{Graphs}_2^\circ).$$

Using the above chain of quasi-isomorphisms, we hence can define a map on cohomology

$$\mathbf{grt} \cong H^1(C) \rightarrow H^1(\text{Def}(G_\infty \rightarrow \text{Graphs}_2^\circ)) \cong H^0(\text{Def}(G_\infty)).$$

As a corollary to Theorem 2 one can prove the following result.

**Proposition 12.** *This composition of maps agrees with D. Tamarkin's map  $\mathbf{grt} \rightarrow H^0(\text{Def}(G_\infty))$  as outlined in section 4.3.1.*

*Proof.* For clarity, we will denote D. Tamarkin's map  $T: \mathbf{grt} \rightarrow H^0(\text{Def}(G_\infty))$  and the map defined just above  $F: \mathbf{grt} \rightarrow H^1(\text{Def}(G_\infty \rightarrow \text{Graphs}_2^\circ))$ . It was shown in Tamarkin's paper that the composition

$$(9) \quad \mathbf{grt} \xrightarrow{T} H^0(\text{Der}(G_\infty)) \rightarrow H^1(\text{Def}(Com_\infty \rightarrow C\mathbf{t})) \cong \mathbf{grt}$$

is the identity. Furthermore, it follows from the definition that the composition

$$\mathbf{grt} \xrightarrow{F} H^1(\text{Def}(G_\infty \rightarrow \text{Graphs}_2^\circ)) \rightarrow H^1(\text{Def}(Com_\infty \rightarrow \text{Graphs}_2^\circ)) \cong \mathbf{grt}$$

is the identity. Denote the following composition by  $T'$ :

$$\mathbf{grt} \xrightarrow{T} H^0(\text{Def}(G_\infty)) \rightarrow H^1(\text{Def}(Com_\infty \rightarrow \text{Graphs}_2^\circ)).$$

Both  $T'$  and  $F$  actually take values in the connected subspace  $H^1(\text{Def}(Com_\infty \rightarrow \text{Graphs}_2^\circ)_{\text{conn}})$ . By Theorem 2 this space is actually isomorphic to  $\mathbf{grt}$  and hence, by what was said above,  $T' = F$ .  $\square$

<sup>7</sup>In our definition of the deformation complex, we use the full cooperad  $e_2^\vee$ , not the version without counit.

**Remark.** One can see that the composition (9) is the identity without using D. Tamarkin's result as follows. By Proposition 14 the map of Theorem 1 agrees with the output of Algorithm 2 in section 6. By Proposition 26 in the Appendix the output of Algorithm 2 agrees with that produced by the composition

$$\mathbf{grt} \xrightarrow{T} H^0(\mathrm{Def}(G_\infty)) \rightarrow H^1(\mathrm{Def}(Com_\infty \rightarrow \mathbf{Graphs}_2^\circ)) \cong H^0(\mathrm{GC}).$$

Composing with  $H^0(\mathrm{GC}) \cong \mathbf{grt}$  again it follows that (9) is the identity.

**Remark.** There is a natural dg Lie algebra structure on  $\mathrm{Def}(G_\infty)$  coming from the commutator of endomorphisms of  $G_\infty$ . By the chain of quasi-isomorphisms above, it follows that there is an  $L_\infty$  structure on  $\mathrm{Def}(G_\infty \rightarrow \mathbf{Graphs}_2)[1]$ , that induces on cohomology the same Lie algebra structure. It is unique up to homotopy, but we cannot write down explicit formulas. Furthermore, by the above proposition, the map  $\mathbf{grt} \rightarrow H^1(\mathrm{Def}(G_\infty \rightarrow \mathbf{Graphs}_2^\circ))$  respects the Lie bracket.

## 5. THE PROOF OF THEOREM 1

In this section we show that  $\mathbf{grt} \cong H^0(\mathrm{GC})$ .<sup>8</sup> By results of the previous sections, we have the following diagram.

$$\begin{array}{ccccc} & & \mathrm{GC} & & \\ & & \downarrow & & \\ \mathbf{grt} & \longrightarrow & \mathrm{Def}(G_\infty \rightarrow \mathbf{Graphs})_{\mathrm{conn}}[1] & \longrightarrow & \mathrm{Def}(Com_\infty \rightarrow \mathbf{Graphs})[1] \end{array}$$

Here the arrows are as follows. The map from  $\mathrm{GC}$  to  $\mathrm{Def}(G_\infty \rightarrow \mathbf{Graphs})_{\mathrm{conn}}$  is given by the action of  $\mathrm{GC}$  on  $\mathbf{Graphs}$ . By Theorem 2 and Proposition 7 it is a quasi-isomorphism in degree 0. The map to  $\mathrm{Def}(Com_\infty \rightarrow \mathbf{Graphs})$  comes from the inclusion  $Com_\infty \hookrightarrow G_\infty$ . The map (defined up to homotopy)  $\mathbf{grt} \rightarrow \mathrm{Def}(G_\infty \rightarrow \mathbf{Graphs})_{\mathrm{conn}}$  was discussed in section 4. There it was also shown that  $H^1(\mathrm{Def}(Com_\infty \rightarrow \mathbf{Graphs})) \cong \mathbf{grt}$  and that the composition of the lower two arrows is the identity on the degree 0 cohomology. It follows that the map

$$\mathbf{grt} \rightarrow H^0(\mathrm{GC})$$

is injective. We next want to show that this map is also surjective, by explicitly showing that the composition of the induced maps on cohomology

$$H^0(\mathrm{GC}) \rightarrow \mathbf{grt} \rightarrow H^0(\mathrm{GC})$$

is the identity. More precisely, we will show that the composition

$$\begin{aligned} H^0(\mathrm{GC}) &\rightarrow H^1(\mathrm{Def}(G_\infty \rightarrow \mathbf{Graphs})_{\mathrm{conn}}) \rightarrow \\ &\rightarrow H^1(\mathrm{Def}(Com_\infty \rightarrow \mathbf{Graphs})) \cong \mathbf{grt} \rightarrow H^1(\mathrm{Def}(G_\infty \rightarrow \mathbf{Graphs})_{\mathrm{conn}}) \end{aligned}$$

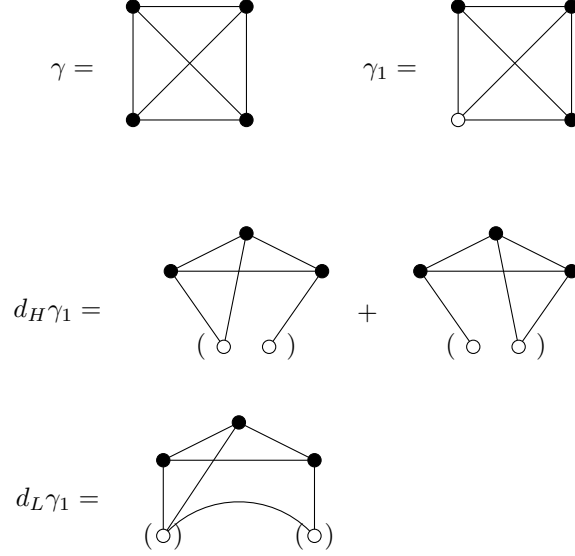
agrees with the map

$$H^0(\mathrm{GC}) \rightarrow H^1(\mathrm{Def}(G_\infty \rightarrow \mathbf{Graphs})_{\mathrm{conn}}).$$

All maps occurring here are given explicitly in earlier sections or in the appendices. Hence it is purely a matter of going through the constructions and checking the above statement. So start with a closed degree 0 element  $\gamma \in \mathrm{GC}$ . Its image in  $\mathrm{Def}(G_\infty \rightarrow \mathbf{Graphs})_{\mathrm{conn}}$  is obtained by acting with  $\gamma$  on the Maurer-Cartan element  $\mu$  in  $\mathrm{Def}(G_\infty \rightarrow \mathbf{Graphs})_{\mathrm{conn}}$ , depicted in Figure 9. To do that, one needs to follow the construction of the action as in Appendix G.3. Note that the formula slightly simplifies since  $\mu$  does not contain internal vertices. From  $\gamma$  one obtains the

<sup>8</sup>We adopt the convention that, if we omit the subscript  $n$  in  $\mathrm{GC}_n$ ,  $\mathbf{Graphs}_n$  etc.,  $n = 2$  is implied. In particular  $\mathrm{GC} := \mathrm{GC}_2$ .

$$(\bigcirc \text{---} \bigcirc) + \begin{array}{c} (\bigcirc) \\ | \\ (\bigcirc) \end{array}$$

FIGURE 9. The Maurer Cartan element  $\mu$  in  $\text{Def}(G_\infty \rightarrow \mathbf{Graphs})_{\text{conn}}$ .FIGURE 10. Illustration of the term  $[\mu, \gamma_1] = d_L\gamma_1 + d_H\gamma_1$ , for  $\gamma$  being the graph corresponding to the **grt**-element  $\sigma_3$ .

element  $\gamma_1 \in \mathbf{Graphs}(1)$  by marking the first vertex as external. We can consider  $\gamma_1$  also as an element of  $\text{Def}(G_\infty \rightarrow \mathbf{Graphs})_{\text{conn}}$ . The action of  $\gamma$  on  $\mu$  is the same as the Lie bracket (in  $\text{Def}(G_\infty \rightarrow \mathbf{Graphs})_{\text{conn}}$ ) of  $\gamma_1$  and  $\mu$ ,  $[\mu, \gamma_1]$ . More explicitly, the latter term splits into two terms corresponding to the two graphs in  $\mu$ , i.e.,  $[\mu, \gamma_1] = d_L\gamma_1 + d_H\gamma_1$ , where we use the notation of Appendix B.1. Concretely, the term  $d_L\gamma_1$  consists of graph with two clusters and  $d_H\gamma_1$  of graphs with one cluster, see Figure 10 for an example. We now have the desired element of  $\text{Def}(G_\infty \rightarrow \mathbf{Graphs})_{\text{conn}}$  and want to map it to  $\text{Def}(Com_\infty \rightarrow \mathbf{Graphs})_{\text{conn}}$ . This is easy, one just picks out those graphs with only one cluster. In our case, we are left with  $d_H\gamma_1$ . Its cohomology class  $[d_H\gamma_1]$  can be interpreted as an element of **grt**. Note that  $d_H\gamma_1$  is in general not exact in  $\text{Def}(Com_\infty \rightarrow \mathbf{Graphs})_{\text{conn}}$ , since  $d_H$  is not the full, but only part of the differential. The full differential is  $d_H + \delta$ , where  $\delta$  denoted the differential coming from that on  $\mathbf{Graphs}$ . We now have a **grt**-element and want to map it to  $\text{Def}(G_\infty \rightarrow \mathbf{Graphs})_{\text{conn}}$  as in Section 4.4. For this, we need to find a representative of  $[d_H\gamma_1]$  in the subcomplex  $C \subset \text{Def}(Com_\infty \rightarrow \mathbf{Graphs}^\circ)_{\text{conn}}$  from Section 4.2. (Reminder: This subcomplex is spanned by graphs with one external vertex and one edge connecting to it.) In our case, such a representative is easily found to be  $-\delta\gamma_1$ . Following Section 4.4 we now have to interpret this element as one in  $\text{Def}(G_\infty \rightarrow \mathbf{Graphs})_{\text{conn}}$ . But the differential on the latter space is  $d_H + d_L + \delta$ , and hence the obtained cohomology class is the same as that represented by  $d_H\gamma_1 + d_L\gamma_1$ . But this was the element we started with and hence we are done.

We still need to show that the maps  $\mathbf{grt} \leftrightarrow H^0(\mathbf{GC})$  respect the Lie algebra structures. We know that the  $\mathbf{grt} \rightarrow H^0(\mathbf{Def}(G_\infty))$  respects Lie brackets (see Section 4.4). It will hence be sufficient to show that the composition

$$H^0(\mathbf{GC}) \rightarrow H^1(\mathbf{Def}(G_\infty \rightarrow \mathbf{Graphs})) \rightarrow H^0(\mathbf{Def}(G_\infty))$$

respects Lie brackets. So let closed degree zero elements  $\gamma, \nu \in \mathbf{GC}$  be given. By the action of  $\mathbf{GC}$  on  $\mathbf{Graphs}$  each closed  $x \in \mathbf{GC}$  of degree zero defines some derivation  $D_x$  of  $\mathbf{Graphs}$ . In particular, we obtain derivations  $D_\gamma, D_\nu$ , satisfying  $[\gamma, \nu] = D_{[\gamma, \nu]}$ . The images of  $\mu, \nu$  in  $\mathbf{Def}(G_\infty \rightarrow \mathbf{Graphs})$  have the form  $D_\mu \circ f, D_\nu \circ f$ , where  $f : G_\infty \rightarrow \mathbf{Graphs}$  is the usual map stemming from the inclusion  $e_2 \rightarrow \mathbf{Graphs}$ . Since the map

$$\phi : \mathbf{Def}(G_\infty) \rightarrow \mathbf{Def}(G_\infty \rightarrow \mathbf{Graphs})[1],$$

is a quasi-isomorphism, there are derivations  $F_\mu, F_\nu, F_{[\mu, \nu]}$  of  $G_\infty$ , such that  $\phi(F_\mu) = D_\mu \circ f + da_\mu$  for some  $a_\mu \in \mathbf{Def}(G_\infty \rightarrow \mathbf{Graphs})$ , and similarly for  $F_\nu$  and  $F_{[\mu, \nu]}$ . Our goal is to show that

$$[F_\mu, F_\nu] = F_{[\mu, \nu]} + (\text{coboundaries}).$$

Since  $\phi$  is a quasi-isomorphism, it is sufficient to show the equation

$$\phi([F_\mu, F_\nu]) := f \circ ([F_\mu, F_\nu]) = f \circ D_{[\mu, \nu]} + (\text{coboundaries}).$$

The commutator on the left is defined such that

$$(id_{G_\infty} + \epsilon F_\mu) \circ (id_{G_\infty} + \epsilon F_\nu) - (\mu \leftrightarrow \nu) = id_{G_\infty} + \epsilon^2 [F_\mu, F_\nu] \quad \text{modulo } \epsilon^3.$$

Applying  $f$  from the left this equation becomes

$$\begin{aligned} & (f + \epsilon D_\mu \circ f) \circ (id_{G_\infty} + \epsilon F_\nu) - (\mu \leftrightarrow \nu) + (\text{coboundaries}) \\ &= (id_{\mathbf{Graphs}} + \epsilon D_\mu) \circ f \circ (id_{G_\infty} + \epsilon F_\nu) - (\mu \leftrightarrow \nu) + (\text{coboundaries}) \\ &= (id_{\mathbf{Graphs}} + \epsilon D_\mu) \circ (id_{\mathbf{Graphs}} + \epsilon D_\nu) \circ f - (\mu \leftrightarrow \nu) + (\text{coboundaries}) \\ &= id_{\mathbf{Graphs}} + \epsilon^2 [D_\mu, D_\nu] + (\text{coboundaries}) \\ &= id_{\mathbf{Graphs}} + \epsilon^2 D_{[\mu, \nu]} + (\text{coboundaries}) \quad \text{modulo } \epsilon^3. \end{aligned}$$

Hence we are done and Theorem 1 has been shown.  $\square$

## 6. EXPLICIT VERSION OF THE MAP(S) BETWEEN $\mathbf{grt}$ AND $\mathbf{GC}_2$

In this section, which is mostly independent from the rest of the paper, we reconstruct the maps from  $\mathbf{GC}_2$  to  $\mathbf{grt}$  and vice versa in an elementary, and more explicit manner. This is not really necessary from a mathematical viewpoint. However, for later applications, it will be good to have as explicit formulas as possible.

Let us start with the map  $H^0(\mathbf{GC}_2) \rightarrow \mathbf{grt}$ . Let a cocycle  $\gamma \in \mathbf{GC}_2$  be given. We want to find a way to read off the  $\mathbf{grt}$  element from the graph. Remember that graphs in  $\gamma$  can be seen as graphs with labelled vertices, invariant under interchange of the labels.

### Algorithm 1:

- (1) We assume that  $\gamma$  is 1-vertex irreducible, which is possible by Proposition 24.
- (2) For each graph in  $\gamma$ , mark the vertex 1 as “external”. This gives a (linear combination of) graph(s)  $\gamma_1 \in \mathbf{Graphs}(1)$ .
- (3) Split the vertex 1 in  $\gamma_1$  into two vertices, redistributing the incoming edges in all possible ways, so that both vertices are hit by at least one edge. Call this linear combination of graphs  $\gamma'_2 \in \mathbf{ICG}_2(2)$ .



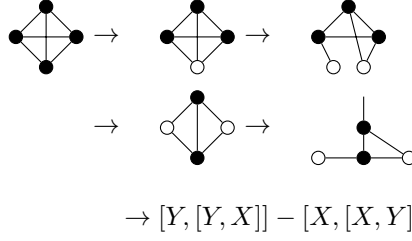


FIGURE 11. Illustration of the algorithm mapping a graph cohomology class to a  $\mathbf{grt}$ -element. Prefactors are omitted. The graphs depicted are (in this order)  $\Gamma$ ,  $\Gamma_1$ ,  $\Gamma'_2$ ,  $\Gamma_2 = T$  and the Lie tree.

- (4)  $\gamma'_2$  is closed in  $\text{ICG}_2$  and has no one-edge component, hence it is the coboundary of some element  $\gamma_2$ . We choose  $\gamma_2$  to be symmetric under interchange of the external vertices 1 and 2.
- (5) Forget the non-internal-trivalent tree part of  $\gamma_2$  to obtain  $T_2$ .
- (6) For each tree  $t$  occurring in  $T_2$  construct a Lie word in (formal) variables  $x, y$  as follows. For each edge incident to vertex 1, cut it and make it the “root” edge. The resulting (internal) tree is a binary tree with leafs labelled by 1 or 2. It can be seen as a Lie tree, and one gets a Lie word  $\phi_1(x, y)$  by replacing each 1 by  $x$  and 2 by  $y$ . Set  $\phi(x, y) = \phi_1(x, y) - \phi_1(y, x)$ . Summing over all such Lie words one gets a (lin. comb. of) Lie word corresponding to  $\gamma$ . Let us call it again  $\phi_\gamma(x, y) \in \mathbb{F}_{\text{Lie}}(x, y)$ .
- (7) The  $\phi_\gamma$  is the desired  $\mathbf{grt}$ -element.

A graphical illustration of this algorithm for the simplest case of a three-wheel (corresponding to  $\sigma_3 \in \mathbf{grt}$ ) is given in Figure 11.

**Proposition 13.** *The above Algorithm 1 produces the correct result, i.e.,  $\phi_\Gamma$  is indeed the  $\mathbf{grt}$ -element corresponding to  $\Gamma$  under the map of Theorem 1.*

*Proof.* We proceed as in the proof of Theorem 1 in the previous section. There we saw that the image of  $\gamma$  in  $\text{Def}(\text{Com}_\infty \rightarrow \text{Graphs}_2)$  is the element  $d_H \gamma_1 = \gamma'_2$ . It can be seen as an element of  $\text{Graphs}_2$ , symmetric under interchange of the external vertices. It furthermore consists entirely of graphs with only one internally connected component by 1-vertex-irreducibility of  $\gamma$ . Now let us compute the element of  $\mathbf{grt}$ , corresponding to the cohomology class of  $d_H \gamma_1$ . First we pick a different representative, which will be  $d_H \gamma_2$ . The latter can be seen as a cocycle in  $\text{ICG}(3)$  satisfying some symmetry property. But any cocycle in  $\text{ICG}(3)$  represents an element of  $\mathfrak{t}_3$ , which can be obtained by restricting to the part consisting of trivalent internal trees. From such a tree one can recover the  $\mathbf{grt}$ -element as described in Appendix F.  $\square$

Next consider the opposite map  $\mathbf{grt} \rightarrow H^0(\text{GC}_2)$ . Let  $\phi \in \mathbf{grt}$ . We can assume without loss of generality that  $\phi$  is homogeneous of degree  $n$  with respect to the grading on  $\mathbf{grt}$ . I.e.,  $\phi = \phi(x, y)$  can be seen as a Lie expression in formal variables  $x, y$ , with  $x, y$  occurring  $n$  times. Consider the complex

$$D := \text{Def}(\text{hoLie}_1 \rightarrow C(\mathfrak{t}))$$

where  $C(\mathfrak{t})$  is the Chevalley complex of the Drinfeld-Kohno (operad of) Lie algebra(s)  $\mathfrak{t}$ , with trivial coefficients.

**Remark.** Since  $C(\mathfrak{t})$  is quasi-isomorphic to  $e_2$  (see [20]), the complex  $D$  has no cohomology in degree zero or lower already by degree reasons. In fact it is acyclic, as can be seen from Proposition 20 in Appendix A.

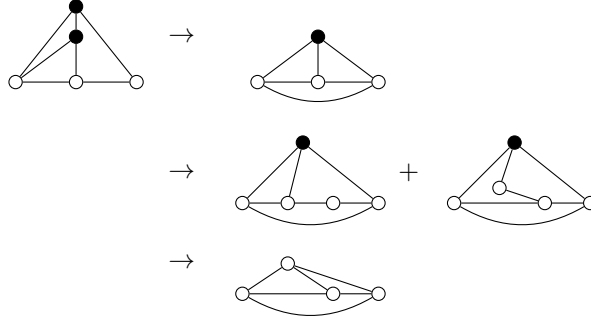


FIGURE 12. Illustration of Algorithm 2, mapping the **gvt**-element  $\sigma_3$  to a graph cohomology class. Prefactors are omitted. The graphs depict (in this order)  $T_3$ ,  $U_3$ ,  $d_{[\cdot]}U_3$ ,  $U_4 = \gamma$ . It is explained in Appendix F how to depict elements of  $C(\mathfrak{t})$  as graphs.

We have the following algorithm:

**Algorithm 2:**

- (1) Symmetrize (and shift in degree) the element  $\phi(t_{12}, t_{23}) \wedge t_{12} \wedge t_{23} \in C(\mathfrak{t}_3)$  so as to obtain an element  $T_3 \in D$ . It is a cocycle (to be shown below).
- (2) Find an element  $U \in D$  with coboundary  $T_3$ . More concretely, split  $U = U_3 + U_4 + \dots + U_{n+1}$  according to the grading on  $D$  by arity in  $\mathfrak{t}$  (i.e., by “number of external vertices” if one thinks in terms of graphs). Then  $d_{CE}U_3 = T_3$ ,  $d_{[\cdot]}U_3 = -d_{CE}U_4$  etc. where  $d_{CE}$  is the part of the differential on  $D$  coming from the Chevalley-Eilenberg differential on  $C(\mathfrak{t})$  and  $d_{CE} + d_{[\cdot]}$  is the full differential.
- (3) By degree reasons,  $U_{n+1} \in C(\mathfrak{t})^{S_{n+1}}[-2n] \subset D$  is a linear combination of wedge products in the generators of  $\mathfrak{t}_{n+1}$  (i.e., the  $t_{ij}$ ’s). Replacing each  $t_{ij}$  by an edge between vertices  $i$  and  $j$  one obtains a linear combination of graphs  $\gamma'$ .
- (4) Drop all graphs in  $\gamma'$  containing vertices of valence smaller than three. This gives some element  $\gamma \in \mathbf{GC}_2$ , the result. It is closed since  $d_{[\cdot]}U_{n+1} = 0$  by construction.

A graphical illustration of this Algorithm for  $\phi = \sigma_3$  can be found in Figure 12.

There are several things to be shown here.

**Lemma 4.** *The element  $T_3$  constructed in the first step is indeed a cocycle.*

*Proof.* First let us show that  $T_3$  is closed under the part  $d_{CE}$  of the differential. Concretely, one has

$$T_3 \propto \phi(t_{12}, t_{23}) \wedge t_{12} \wedge t_{23} + \phi(t_{23}, t_{31}) \wedge t_{23} \wedge t_{31} + \phi(t_{31}, t_{12}) \wedge t_{31} \wedge t_{12}$$

and hence

$$\begin{aligned} d_{CE}T_3 &\propto -\phi(t_{12}, t_{23}) \wedge [t_{12}, t_{23}] + (\text{cycl.}) \\ &\quad + [\phi(t_{12}, t_{23}), t_{12}] \wedge t_{23} - [\phi(t_{12}, t_{23}), t_{23}] \wedge t_{12} + (\text{cycl.}) \\ &= 0 + ([\phi(t_{12}, t_{23}), t_{12}] - [\phi(t_{23}, t_{31}), t_{31}]) \wedge t_{23} + (\text{cycl.}) \\ &= 0 \end{aligned}$$

Here the term in the first line on the right is zero because of the symmetries of  $\phi(t_{12}, t_{23})$ . In more detail,  $[t_{12}, t_{23}]$  is antisymmetric under the  $S_3$  action on indices.

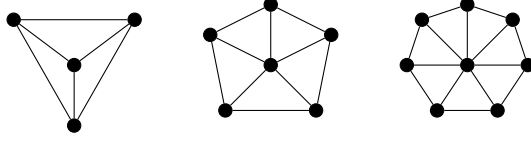


FIGURE 13. Some wheel graphs. The element corresponding to a generator  $\sigma_{2j+1}$  of  $\mathbf{grt}$  contains such a wheel with  $2j + 1$  spokes.

Hence the symmetrization of  $\phi(t_{12}, t_{23}) \wedge [t_{12}, t_{23}]$  picks out the antisymmetric part of  $\phi(t_{12}, t_{23})$ , which is zero by the hexagon equation. The second equality follows directly from the cabling relation (or equivalently, the semiclassical hexagon), which in turn follows from the hexagon equation (see [2] for a proof and more details on those relations).

Next we need to show that  $T_3$  is also closed under  $d_{[\cdot, \cdot]}$ . This is tedious if one writes everything out in detail. Gathering similar terms one obtains

$$d_{[\cdot, \cdot]} T_3 \propto (A) \wedge t_{12} \wedge t_{23} \wedge t_{34} + (B) \wedge t_{12} \wedge t_{23} \wedge t_{24} + (\dots)$$

where the terms  $(\dots)$  can be obtained from the first two terms by permutations of indices, so as to make whole expression symmetric. One calculates

$$A = -\phi^{2,3,4} - \phi^{1,2,3} + \phi^{12,3,4} - \phi^{1,23,4} + \phi^{1,2,34} = 0$$

by the pentagon equation. Here we use the usual notation  $\phi^{12,3,4} = \phi(t_{13} + t_{23}, t_{34})$  etc. Similarly one computes

$$\begin{aligned} B &= -\phi^{1,2,3} - \phi^{3,2,4} + \phi^{1,2,4} + \phi^{1,24,3} - \phi^{1,23,4} + \phi^{3,12,4} \\ &= -\phi^{1,2,3} - \phi^{3,2,4} - \phi^{2,4,3} + \phi^{12,3,4} - \phi^{1,23,4} + \phi^{1,2,34} \\ &= -\phi^{3,2,4} - \phi^{2,4,3} + \phi^{2,3,4} = 0. \end{aligned}$$

Here we used twice the pentagon and once the hexagon equation. Hence  $d_{[\cdot, \cdot]} T_3 = 0$ .  $\square$

Now that we know that  $T_3$  is closed, we can construct  $U$ , which exists and is unique up to exact terms by the remark preceding Algorithm 2.

**Proposition 14.** *The cohomology class represented by the output  $\gamma$  of the above Algorithm 2 indeed is the image of the  $\mathbf{grt}$ -element  $\phi$  under the map of Theorem 1.*

*Proof.* The proof of Proposition 16 uses the above algorithm. Hence it shows that the composition  $H^0(\mathbf{GC}) \rightarrow \mathbf{grt} \rightarrow H^0(\mathbf{GC})$  is the identity, where the right arrow is the algorithm, and the left arrow is the map from Theorem 1 (or, equivalently by Proposition 13, the map computed with the first algorithm). The proposition follows.  $\square$

### 6.1. Explicit form of the generators.

**Proposition 15.** *Under the map of Theorem 1, the element  $\sigma_{2j+1} \in \mathbf{grt}$  corresponds to a graph cohomology class, all of whose representatives have a nonvanishing coefficient in front of the wheel graph with  $2j + 1$  spokes, see Figure 13.*

There are (at least) two proofs: One can trace through any of the two algorithms introduced above and see that the wheel must be present. All we need to know (and, sadly, almost all we do know) about  $\sigma_{2j+1}$  is that it contains a term

$$\mathrm{ad}_x^{2j}(y).$$

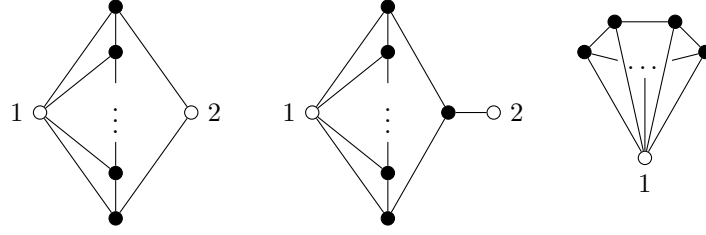


FIGURE 14. Three graphs occuring in the first proof of Proposition 15.

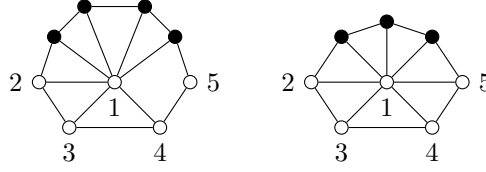


FIGURE 15. Pictures of the elements  $x_2$  (left) and  $y_3$  (right) for the element  $\sigma_7 \in \mathfrak{grt}$ . We use here the identification of  $C(\mathfrak{t})$  with graphs (see Appendix F) and depict only the part of lowest valence at vertex 5.

*First proof (using first algorithm).* In order to produce a term  $\text{ad}_x^{2j}(y)$  at the end, the  $T_2$  in the algorithm, and hence also the  $\gamma_2$ , has to contain a graph as depicted in Figure 14 (left). But this means that in  $\gamma'_2$  there is a term of the form depicted in Figure 14 (middle). But such a term can only be produced if  $\gamma_1$  contains a term like that in Figure 14 (right). But this means that  $\gamma$  has to contain a wheel graph.  $\square$

*Second proof (using second algorithm).* We do not care about signs and prefactors. In  $T_3$  one has a term  $x_0 := \text{ad}_{t_{12}}^{2j}(t_{23}) \wedge t_{12} \wedge t_{23}$ . The only term that can produce this under the Chevalley differential is  $y_1 := \text{ad}_{t_{12}}^{2j-1}(t_{23}) \wedge t_{12} \wedge t_{23} \wedge t_{13}$ . Hence  $U_3$  must contain such a term. Hence  $d_{[\cdot]}U_3$  contains (among others) the term

$$x_1 := \text{ad}_{t_{12}}^{2j-1}(t_{24}) \wedge t_{12} \wedge t_{13} \wedge t_{23} \wedge t_{34}.$$

The term  $y_1$  in  $U_3$  we considered is furthermore the only one that can produce  $x_1$ , hence the coefficient in front of  $x_1$  in  $d_{[\cdot]}U_3$  is nonzero. The term  $x_1$  can only be produced by the Chevalley differential from the term

$$y_2 := \text{ad}_{t_{12}}^{2j-2}(t_{24}) \wedge t_{12} \wedge t_{13} \wedge t_{14} \wedge t_{23} \wedge t_{34}$$

and hence this must be contained in  $U_4$ . Computing  $d_{[\cdot]}U_4$  we get a term with nonzero coefficient

$$x_2 := \text{ad}_{t_{12}}^{2j-2}(t_{25}) \wedge t_{12} \wedge t_{13} \wedge t_{14} \wedge t_{23} \wedge t_{34} \wedge t_{45}.$$

It can only be produced from

$$y_3 := \text{ad}_{t_{12}}^{2j-3}(t_{25}) \wedge t_{12} \wedge t_{13} \wedge t_{14} \wedge t_{15} \wedge t_{23} \wedge t_{34} \wedge t_{45}$$

etc. At the end one sees that  $U_{2j+2}$  must contain a term

$$y_{2j} := \text{ad}_{t_{12}}^0(t_{2(2j+1)}) \wedge t_{12} \wedge t_{13} \wedge \cdots \wedge t_{1(2j+1)} \wedge t_{15} \wedge t_{23} \wedge t_{34} \wedge \cdots \wedge t_{(2j+1)2}.$$

This is the wheel.

In Figure 15 one finds a graphical illustration of the elements  $y_k$  and  $x_k$  occuring in this proof.  $\square$

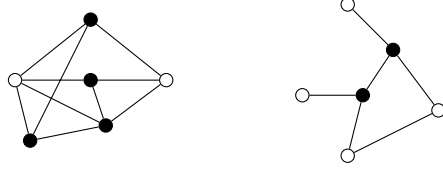


FIGURE 16. Graphs in the complex DGC.

**Remark.** The second proof does not use Theorem 1. Hence, noticing that a graph cocycle containing a wheel is never exact, it shows independently that there is a nontrivial cohomology class in  $H^0(\text{GC})$  for every odd number  $\geq 3$ . This class contains a wheel. On the other hand, it is easy to see that each such class is mapped to an element  $\sigma_{2j+1} \in \mathfrak{grt}$  (up to commutators) under the map  $H^0(\text{GC}) \rightarrow \mathfrak{grt}$ , i.e., using the first algorithm. If we assume the Deligne-Drinfeld conjecture, this gives an independent “proof” that the map  $\mathfrak{grt} \rightarrow H^0(\text{GC})$  is injective, and the map  $H^0(\text{GC}) \rightarrow \mathfrak{grt}$  surjective. One can upgrade this reasoning to a more or less elementary “proof” (modulo the Deligne-Drinfeld conjecture) of Theorem 1, as we will do in the next section.

**6.2. A second, elementary “proof” of Theorem 1, modulo the Deligne-Drinfeld conjecture.** By Remark 6.1 one obtains such a “proof” if one can show the following Proposition by elementary means.

**Proposition 16.** *The composition of the maps  $H^0(\text{GC}) \rightarrow \mathfrak{grt} \rightarrow H^0(\text{GC})$  is the identity.*

The central role in the proof will be played by the (double) complex with underlying vector space

$$\text{DGC} := \text{Def}(\text{hoLie}_1 \rightarrow \text{Graphs}_2)_{\text{conn}} \oplus \text{GC}.$$

It is spanned by graphs with “unidentifiable” vertices in two colors, say black and white, see Figure 16. The differential has 3 parts: (i) the usual differential on  $\text{Def}(\text{hoLie}_1 \rightarrow \text{Graphs}_2)$  (ii) the usual differential on  $\text{GC}$  (iii) a part from  $\text{GC}$  to  $\text{Def}(\text{hoLie}_1 \rightarrow \text{Graphs}_2)_{\text{conn}}$  given on a graph by adding a white vertex and connecting it with a single edge to some black vertex. This part is depicted in Figure 18.

More precisely, there is a Lie bracket on  $\text{DGC}$ , defined similarly as for the usual graph complex:

$$\begin{aligned} [\Gamma_1 + \Gamma_2, \tilde{\Gamma}_1 + \tilde{\Gamma}_2] &= [\Gamma_1, \tilde{\Gamma}_1]_{\text{Def}(\text{hoLie}_1 \rightarrow \text{Graphs}_2)} + \\ &\quad + [\Gamma_2, \tilde{\Gamma}_2]_{\text{GC}} + \Gamma_1 \circ_b \tilde{\Gamma}_2 - (-1)^{|\Gamma_2||\tilde{\Gamma}_1|} \tilde{\Gamma}_1 \circ_b \Gamma_2 \end{aligned}$$

where  $\Gamma_1, \tilde{\Gamma}_1 \in \text{Def}(\text{hoLie}_1 \rightarrow \text{Graphs}_2)$  and  $\Gamma_2, \tilde{\Gamma}_2 \in \text{GC}$ , and  $\circ_b$  is the insertion at black vertices. Note the antisymmetry in black and white vertices.

**Remark.** The complex  $\text{DGC}$  can be seen as the universal version of a subcomplex<sup>9</sup> of the Chevalley complex of the Lie algebra  $T_{\text{poly}} \ltimes T_{\text{poly}}$  where the first factor acts on the second by the adjoint action. The subcomplex is given by those chains that take values in the first factor iff all inputs are in the first factor. We will denote this subcomplex  $C'(T_{\text{poly}} \ltimes T_{\text{poly}})$ . It is isomorphic to the complex  $C(T_{\text{poly}} \ltimes T_{\text{poly}}, T_{\text{poly}})$ , but carries a natural Lie algebra structure. In graphs, black vertices correspond to

<sup>9</sup>...and sub-dg Lie algebra

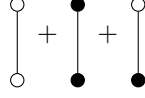


FIGURE 17. The MC element giving the complex DGC its differential.

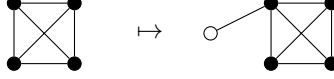


FIGURE 18. One part of the differential on DGC. The factor 4 arises because of the symmetries of the graph.

the first factor, whites to the second. Elements of  $\mathbf{GC}$  are considered as cochains with values in the first factor, elements of  $\mathbf{Def}(hoLie_1 \rightarrow \mathbf{Graphs}_2)_{conn}$  as cochains with values in the second.

The differential can be seen as the bracket with the Maurer-Cartan element depicted in Figure 17, although this element does not lie in DGC, only in some bigger Lie algebra obtained by dropping the trivalence condition.

The important fact about DGC is that there are maps (of dg Lie algebras)

$$(10) \quad \mathbf{GC} \rightarrow \mathbf{DGC} \rightrightarrows \mathbf{GC}.$$

One of the right hand arrows is the obvious projection to the  $\mathbf{GC}$  part, one is the projection to (at least) trivalent graphs with only white vertices. The left hand arrow is more interesting. It send a graph to the sum of graphs obtained by coloring its vertices black or white in all possible ways.

**Remark.** Interpret DGC again as universal version of  $C'(T_{poly} \ltimes T_{poly})$ . One has a map of Lie algebras  $T_{poly} \ltimes T_{poly} \rightarrow T_{poly}$  which is the identity on both summands. Hence one has a (composition) map

$$C(T_{poly}) \rightarrow C(T_{poly} \ltimes T_{poly}, T_{poly}) \rightarrow C'(T_{poly} \ltimes T_{poly})$$

which one checks to be compatible with the (pre-)Lie product. The map  $\mathbf{GC} \rightarrow \mathbf{DGC}$  is just the universal version of this map.

It is easily seen that both compositions of the maps (10) are the identity. Furthermore, the maps are quasi-isomorphisms. The cohomology can be computed by taking a spectral sequence, such that the first differential is the one producing new black vertices. The first convergent is then (as vector space)

$$\mathbf{Def}(hoLie_1 \rightarrow Lie) \oplus H(\mathbf{GC}).$$

The next differential is the usual differential on the first summand, and zero on the second.<sup>10</sup> But the first summand is acyclic, as we already saw.

*Sketch of proof of Proposition 16.* First let us make a small adaptation of the algorithm computing the map  $\mathbf{grt} \rightarrow H^0(\mathbf{GC})$ . We worked originally with the complex

$$D := \mathbf{Def}(hoLie_1 \rightarrow C(\mathfrak{t}))_{conn}.$$

Let us change this complex to

$$D' := \mathbf{Def}(hoLie_1 \rightarrow C(\mathbf{ICG}))_{conn} = \mathbf{Def}(hoLie_1 \rightarrow \mathbf{Graphs})_{conn} \subset \mathbf{DGC}.$$

<sup>10</sup>The latter statement is seen as follows: Graph in  $\mathbf{Def}(hoLie_1 \rightarrow Lie)$  have  $k$  vertices and  $k - 1$  edges. The differential of some graph with  $k - 1$  vertices in  $\mathbf{GC}$  must have  $k$  vertices and at least  $1 + \frac{3}{2}(k - 1) = \frac{3}{2}k - \frac{1}{2} > k - 1$  edges, hence the differential must be zero.

This does not matter since  $H(\mathrm{ICG}) \cong \mathfrak{t}$ .

Let now  $\Gamma \in \mathbf{GC}$  be a degree 0 cocycle, say homogeneous of degree  $n$  wrt. the grading by number of vertices. Let  $\Gamma_0 + \Gamma_1 + \dots + \Gamma_n \in \mathbf{DGC}$  be the image of  $\Gamma$ , with  $\Gamma_j$  being the part with  $j$  external (white) vertices. Split the differential on  $\mathbf{DGC}$  into  $d_0 + d_1$  where  $d_0$  does not increase the number of external vertices. In particular  $d_0\Gamma_{j+1} + d_1\Gamma_j = 0$ . We call some graph internally connected if it is connected after deleting all white vertices. Note: Since  $\Gamma$  is 1-vertex irreducible,  $\Gamma_1$  is internally connected.  $d_1\Gamma_1 = d_0\Gamma_2$  hence has two internally connected components ( $=:\text{i.c.c.}$ ), one of which is a single edge. One can check using results of [18] that it follows that there is some  $X_2 \in \mathbf{DGC}$  with two external vertices, such that  $\Gamma_2 + d_0X_2$  has at most two i.c.c., and the part with two i.c.c.s has an edge as one of them. Removing the edge, one obtains the  $\Gamma_2$  of the first algorithm, which determines the  $\mathbf{grt}$ -element.

Next look at the second algorithm, taking that  $\mathbf{grt}$ -element and producing some graph cohomology class. It turns out that for  $T_3$  one can equivalently take the element  $T_3 = d_1(\Gamma_2 + d_0X_2)$ . For the element  $U$  we can then take

$$U = \Gamma_3 - d_1X_2 + \Gamma_4 + \dots + \Gamma_n.$$

which satisfies  $(d_0 + d_1)U = T_3$ . The graph cohomology class is given by the part  $U_n = \Gamma_n$ , which is the original cochain  $\Gamma$  we started with.  $\square$

## 7. APPLICATION: ACTION ON DEFORMATION QUANTIZATION

We suppose in this section that the reader is already somewhat familiar with the basic objects and questions of deformation quantization. If not, we refer to M. Kontsevich's paper [15].

**7.1. The action.** Let  $T_{\text{poly}}^\bullet = \Gamma(\mathbb{R}^n; \wedge^\bullet T\mathbb{R}^n)$  be the space of multivector fields on  $\mathbb{R}^n$ . It is a Gerstenhaber algebra with the wedge product and the Schouten bracket. There is an action of the operad  $\mathbf{Gra}_2$  on  $T_{\text{poly}}$ , considered as a vector space. Concretely, for a graph  $\Gamma \in \mathbf{Gra}_2(N)$  and  $\gamma_1, \dots, \gamma_N \in T_{\text{poly}}$  the action reads

$$\Gamma(\gamma_1, \dots, \gamma_N) = \mu \circ \left( \prod_{(i,j)} \sum_{k=1}^d \frac{\partial}{\partial x_k^{(j)}} \frac{\partial}{\partial \xi_k^{(i)}} + \frac{\partial}{\partial x_k^{(i)}} \frac{\partial}{\partial \xi_k^{(j)}} \right) (\gamma_1 \otimes \dots \otimes \gamma_n).$$

Here  $\mu$  is the operation of multiplication of  $n$  multivector fields and the product runs over all edges  $(i, j)$  in  $\Gamma$ , in the order given by the numbering of edges. The notation  $\frac{\partial}{\partial x_k^{(j)}}$  means that the partial derivative is to be applied to the  $j$ -th factor of the tensor product, and similarly for  $\frac{\partial}{\partial \xi_k^{(i)}}$ . The Gerstenhaber algebra structure on  $T_{\text{poly}}$  can be recovered from this action via the operad map  $e_2 \rightarrow \mathbf{Gra}_2$  sending the product to the empty graph and the bracket to the graph with two vertices and one edge. It also follows that there is an action of the dg Lie algebra  $\text{Der}(L_\infty \rightarrow \mathbf{Gra}_2)$  on  $T_{\text{poly}}$  by pre- $L_\infty$ -derivations. Here the closed degree zero elements act by true  $L_\infty$ -derivations. In particular, the closed degree zero elements of  $\mathbf{GC}_2 \subset \text{Der}(L_\infty \rightarrow \mathbf{Gra}_2)$  act in this way. By the identification  $\mathbf{grt} \cong H^0(\mathbf{GC}_2)$  there is also an action of  $\mathbf{grt}$  on  $T_{\text{poly}}$ , defined up to homotopy.

Any such action on  $T_{\text{poly}}$  also yields an action on the space of its Maurer-Cartan elements, i.e., on Poisson structures. This was one of the original motivations of M. Kontsevich, see [13].

Furthermore, let  $D_{\text{poly}}$  be the space of polydifferential operators on  $\mathbb{R}^n$ . The main result of deformation quantization is M. Kontsevich's Formality Theorem, stating the existence of an  $L_\infty$  quasi-isomorphism

$$T_{\text{poly}} \rightarrow D_{\text{poly}}.$$

In the following, such a morphism will be called a formality morphism. By composition,  $\text{Der}(L_\infty \rightarrow \text{Gra}_2)$ ,  $\text{GC}_2$  and  $\mathbf{grt}$  act also on the space of formality morphisms.

**7.2. Comparison with D. Tamarkin’s  $\mathbf{grt}$ -action on  $T_{\text{poly}}$ .** There is the following result.

**Proposition 17.** *D. Tamarkin’s  $\mathbf{grt}$ -action up to homotopy on  $T_{\text{poly}}$  agrees with the pullback of the natural  $\text{GC}$ -action under the map  $\mathbf{grt} \rightarrow H^0(\text{GC})$  from Theorem 1.*

*Proof.* D. Tamarkin’s action factors through  $H^0(\text{Def}(G_\infty))_{\text{conn}} \cong H^0(\text{GC})$ , and D. Tamarkin’s map  $\mathbf{grt} \rightarrow H^0(\text{Def}(G_\infty))_{\text{conn}} \cong H^0(\text{GC})$  is the one from Theorem 1.  $\square$

**7.3. Action on formality morphisms.** Let  $\Phi$  be a Drinfeld associator, i.e., a power series in some variables  $x, y$ , which is group-like and satisfies the pentagon, hexagon and antisymmetry conditions as in Section 4. To  $\Phi$  one can associate a formality morphism

$$\mathcal{U}_\Phi: T_{\text{poly}} \rightarrow D_{\text{poly}}.$$

In fact, there are multiple, conjecturally equivalent ways to associate the morphism  $\mathcal{U}_\Phi$  to the Drinfeld associator  $\Phi$ . Let us choose here D. Tamarkin’s construction using formality of the little disks operad. In any way, one obtains many inequivalent formality morphisms, one for each associator. On the other hand, M. Kontsevich’s original proof featured a very different construction of one particular formality morphism  $\mathcal{U}_K$ , using graphical techniques. This construction contains essentially no “free parameters”. It has been an open question for some time how D. Tamarkin’s and M. Kontsevich’s constructions of Formality Morphisms fit together.

I think I can now give an answer, to some extend. First it is shown in [22] that in D. Tamarkin’s formality morphism is homotopic to the one constructed by M. Kontsevich [15], if one chooses for the associator the Alekseev-Torossian associator  $\Phi_{AT}$ . Secondly, there is an action of  $\mathbf{grt}$  on the set of Drinfeld associators according to the formula

$$(\psi \cdot \Phi)(x, y) = \left. \frac{d}{dt} \right|_{t=0} \exp(t\psi) \Phi(\exp(-t\psi)x \exp(t\psi), y)$$

for  $\psi \in \mathbf{grt}$ . This action is free and transitive. Hence any Drinfeld associator can be obtained from one particular, say  $\Phi_{AT}$ , by integrating the flow on the space of Drinfeld associators generated by some  $\mathbf{grt}$ -element. By this action  $\mathbf{grt}$  also acts on formality morphisms of the form  $\mathcal{U}_\Phi$ . This action is the same as the one we previously encountered, up to homotopy. Fix a  $\mathbf{grt}$ -element  $\psi$ . It can be mapped to graph cohomology class, represented by some  $\Gamma \in \text{GC}_2$ . We saw earlier that the actions of  $\psi$  and  $\Gamma$  on  $T_{\text{poly}}$  agreed, up to homotopy. From these arguments, one see that any formality morphism of the form  $\mathcal{U}_\Phi$  can be obtained, up to homotopy, by integrating the action of some cocycle  $\Gamma \in \text{GC}_2$  on  $T_{\text{poly}}$ , followed by applying Kontsevich’s formality morphism  $\mathcal{U}_K$ . Since  $\mathcal{U}_K$  is already constructed by “graphical means”, one can incorporate this action into the construction of  $\mathcal{U}_K$ . One just has to modify the way weights are associated to graphs in M. Kontsevich’s definition of  $\mathcal{U}_K$ . This shows how one can obtain, using “purely graphical techniques”, a big class of formality morphisms. More details and a discussion of a related construction of Drinfeld associators using configuration space integrals will be contained in forthcoming paper.



**7.4. Answer to a question of B. Tsygan.** Let  $\Phi$  again be a Drinfeld associator. To  $\Phi$ , one can assign a formal odd power series

$$f_{\Phi}(x) = \sum_{j \geq 1} f_{2j+1} x^{2j+1}.$$

The number  $f_{2j+1}$  is the coefficient of  $x^{2j}y$  in the series  $\Phi(x, y)$ . Now let  $\psi$  be a  $\mathbf{grt}$ -element, and use it to act on  $\Phi$  as in the previous section. It is easy to check that this action changes the coefficient of  $x^{2j}y$  by the coefficient of  $\mathrm{ad}_x^{2j}y$  in  $\psi$ . Let us call this coefficient  $\tilde{s}_{2j+1}(\psi)$ . The  $\tilde{s}_{2j+1}$  ( $j = 1, 2, \dots$ ) can be checked to vanish on  $[\mathbf{grt}, \mathbf{grt}]$  and hence form Lie algebra cocycles.

Next let  $\mathcal{U}_{\Phi}$  be the formality morphism associated to  $\Phi$  as in the previous subsection. One can use it to construct a proof of Duflo's Theorem along the lines of [15], section 8. The Duflo morphism has the form

$$J \circ e^{\sum_{j \geq 1} \tilde{f}_{2j+1} \mathrm{tr}(\mathrm{ad}_{\partial}^{2j+1})}$$

where  $J$  is the usual Duflo morphism. Hence there is a natural way to define another odd formal power series for  $\Phi$ , namely

$$\tilde{f}_{\Phi} = \sum_{j \geq 1} \tilde{f}_{2j+1} x^{2j+1},$$

B. Tsygan asked the following question: Is  $f_{\Phi} = \tilde{f}_{\Phi}$ ?

**Lemma 5.** *The answer to B. Tsygan's question is yes.*

Since the action of  $\mathbf{grt}$  on associators is transitive, it will be sufficient to prove the following two statements.

- (1) For the Alekseev-Torossian associator  $\Phi_{AT}$ ,  $f_{\Phi_{AT}} = \tilde{f}_{\Phi_{AT}}$ .
- (2) For  $\psi \in \mathbf{grt}$  and any associator  $\Phi$ ,

$$f_{\psi \cdot \Phi} = \tilde{f}_{\psi \cdot \Phi}$$

where  $f_{\psi \cdot \Phi}$  should be understood as “derivative of  $f_{(\cdot)}$  along  $\psi \cdot$  at  $\Phi$ ”.

Let us begin with the first statement. The Alekseev-Torossian associator can be shown to be even, i.e., it contains only monomials in  $x, y$  of even degree. Hence  $f_{\Phi_{AT}} = 0$ . From the previous subsection on the other hand we know that  $\mathcal{U}_{\Phi_{AT}}$  is homotopic to Kontsevich's morphism  $\mathcal{U}_K$ . It is been shown in [15] and [19] that from  $\mathcal{U}_K$  one obtains the original Duflo morphism. Hence we conclude  $\tilde{f}_{\Phi_{AT}} = 0 = f_{\Phi_{AT}}$ .

For the second statement, fix some  $\psi \in \mathbf{grt}$ . The left hand side of the equation we want to show is  $f_{\psi \cdot \Phi} = \sum_j \tilde{s}_{2j+1}(\psi) x^{2j+1}$ . Let  $\Gamma$  be graph cocycle, whose cohomology class corresponds to  $\psi$ . From the identification of Lie algebras  $\mathbf{grt} \cong H^0(\mathrm{GC}_2)$  one (of course) also obtains an identification of the Lie algebra cohomology classes. The class represented by  $\tilde{s}_{2j+1}$  corresponds to the graph homology class represented by a wheel with  $2j+1$  spokes, as can be seen from the proofs in Section 6.1. If we call the graph cycle given by this wheel graph  $s_{2j+1} \in \mathrm{GC}_2^*$ , we can hence write

$$f_{\psi \cdot \Phi} = \sum_j s_{2j+1}(\Gamma) x^{2j+1}.$$

Next we should compute  $\tilde{f}_{\psi \cdot \Phi}$ . Suppose we change our formality morphism by precomposing it with some  $L_{\infty}$ -automorphism  $\mathcal{V}$  of  $T_{\mathrm{poly}}$ . Then the associated Duflo morphism changes by precomposing with the automorphism  $\mathcal{V}_1^{\pi}$  of  $S\mathfrak{g}$ . Here  $\mathfrak{g}$  is the Lie algebra for which we write down the Duflo morphism and  $\mathcal{V}_1^{\pi}$  is the first Taylor component of the  $L_{\infty}$ -morphism obtained by twisting  $\mathcal{V}$  by the Poisson structure  $\pi$  on  $\mathfrak{g}^*$ . Suppose now that that  $\mathcal{V}$  is obtained by integrating the action of the graph cocycle  $\Gamma$ , i.e.,  $\mathcal{V} = \exp(t\Gamma \cdot)$ . We can assume without loss of generality

that  $\Gamma$  is 1-vertex irreducible. Then using the linearity of  $\pi$  it can be checked that the only terms in  $\mathcal{V}_1^\pi$  that do not vanish on  $S\mathfrak{g}$  come from the wheel graphs. More precisely

$$\mathcal{V}_1^\pi|_{S\mathfrak{g}} = e^{t \sum_j s_{2j+1}(\Gamma) \text{tr}(\text{ad}_{\mathfrak{g}}^{2j+1})}.$$

Taking the derivative at  $t = 0$  we can hence conclude that  $\tilde{f}_{\psi \cdot \Phi} = \sum_j s_{2j+1}(\Gamma) x^{2j+1} = f_{\psi \cdot \Phi}$ .  $\square$

**7.5. Globalization.** Degree zero cocycles of  $\text{GC}_2$  can be naturally interpreted as  $L_\infty$ -derivations of the polyvector fields on  $\mathbb{R}^n$ . Using the standard globalization methods as in [6, 5, 3], one can obtain  $L_\infty$ -derivations also on the space of polyvector fields on any smooth manifold, or on the sheaf cohomology of the sheaf of holomorphic polyvector fields on a complex manifold. The latter case seems more interesting. By a small computation, done jointly with V. Dolgushev, one can see that the cocycles corresponding to the  $\sigma_{2j+1}$  in this case act by contraction with the odd Chern characters. The details will be described in a separate paper.

## 8. APPLICATION: CHECKING THE DELIGNE-DRINFELD CONJECTURE IN LOW ORDERS

Consider the conjectural generators  $\sigma_j$  (for odd  $j \geq 3$ ) of  $\mathfrak{grt}$ . It is clear that the set of  $\sigma_j$ 's is linearly independent because of the grading on  $\mathfrak{grt}$  by “number of  $t_{ij}$ 's”. Next consider the commutators

$$\sigma_{ij} = [\sigma_i, \sigma_j].$$

Fix  $N := i + j$ . Then the  $\sigma_{ij}$  span a vector space

$$V_N := \text{span}\{\sigma_{ij} \mid i > j \geq 3, i, j \text{ odd}, i + j = N\}.$$

The Deligne-Drinfeld conjecture asserts that the  $\sigma_{ij}$  for  $i > j \geq 3$  are linearly independent and hence conjecturally

$$\dim V_N = \lfloor \frac{1}{4}N - 1 \rfloor \quad (\text{conjecture}).$$

The main result of this section is the following lower bound.

**Proposition 18.** *Let  $N$  be an even positive integer. The space  $V_N \subset \mathfrak{grt}$  spanned by  $\{\sigma_{ij} \mid i > j \geq 3, i, j \text{ odd}, i + j = N\}$  is at least  $\lceil \frac{1}{6}N - 1 \rceil$ -dimensional.*

The proof proceeds by explicitly constructing a series of homology classes in the dual of the graph complex. The graph complex  $\text{GC}$  splits into a direct sum of finite dimensional sub-complexes

$$\text{GC} = \bigoplus_\alpha \text{GC}^{(\alpha)}$$

as discussed in Appendix E. We define the graded dual of  $\text{GC}$  as the complexes

$$\text{GC}^* := \bigoplus_\alpha (\text{GC}^{(\alpha)})^*.$$

Recall that elements of  $\text{GC}$  are linear combinations of graphs (with numbered vertices), invariant under permutations of the vertex labels. Similarly, elements of  $\text{GC}^*$  should be thought of as linear combinations of graphs with numbered vertices, modulo permutations of the vertex labels. It is clear that there is a natural pairing

$$\langle \cdot, \cdot \rangle : \text{GC}^* \times \text{GC} \rightarrow \mathbb{R}.$$

From this and the dg Lie algebra structure on  $\text{GC}$ , one obtains a dg Lie coalgebra structure on  $\text{GC}^*$ . Concretely, the differential on  $\text{GC}^*$  is given by edge contraction.

It is possible to compute a series of degree zero homology classes of  $\text{GC}^*$ . By using the pairing  $\text{GC}^* \times \text{GC} \rightarrow \mathbb{R}$  one can extract from this series linear independence properties of classes in  $H^0(\text{GC}) \cong \mathfrak{grt}$ .

Let us start with the simplest (and admittedly trivial) example. Let  $\sigma_j$ , for odd  $j \geq 3$ , by abuse of notation also denote representatives in  $\mathbf{GC}$  of the cohomology classes corresponding to the conjectural generators of  $\mathbf{grt}$ , see Section 6.1 for details. By Proposition 15  $\sigma_j$  contains a term corresponding to a wheel-graph with  $j$  spokes as in Figure 13. Since the wheel graph is not closed for  $j > 3$ ,  $\sigma_j$  will in general also contain other terms. However, dually in  $\mathbf{GC}^*$ , all wheel graphs are closed and hence define homology classes. Let us denote by  $s_j \in \mathbf{GC}^*$ , for odd  $j \geq 3$ , the wheel graph with  $j$  spokes. It is a priori not clear that the  $s_j$  represent nontrivial homology classes. However, if we normalize correctly, we know that

$$\langle s_i, \sigma_j \rangle = \delta_{ij}.$$

From this statement one can immediately conclude that

- (1) The homology classes  $\{[s_j] \mid j = 3, 5, \dots\}$  are linearly independent.
- (2) The cohomology classes  $\{[\sigma_j] \mid j = 3, 5, \dots\}$  are linearly independent.

Let us go further and look at the first commutators  $\sigma_{ij} := [\sigma_i, \sigma_j] = -\sigma_{ji}$ .<sup>11</sup> Now suppose that we had cycles  $s_{ij} = -s_{ji} \in \mathbf{GC}^*$  such that

$$\langle s_{ij}, \sigma_{kl} \rangle = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}.$$

Then we could conclude as above that the cohomology classes  $[\sigma_{ij}] \in H^0(\mathbf{GC})$  ( $i, j = 3, 5, \dots$  and  $i < j$ ) and the homology classes  $[s_{ij}] \in H^0(\mathbf{GC}^*)$  ( $i, j = 3, 5, \dots$  and  $i < j$ ) both form linearly independent families. Unfortunately, I cannot write down all the  $s_{ij}$  or arbitrary odd  $i$  and  $j$ . However, I can write down formulas for a smaller series of homology classes  $t_{pq}$ , where  $p > q > 0$  and  $q$  is even. Each  $t_{pq}$  will be a linear combination of graphs with  $2p + q + 1$  vertices. Deferring the definition, the  $t_{pq}$  will satisfy the following properties.

**Proposition 19.** (1) *The homology classes  $\{[t_{pq}] \mid p > q > 0, q \text{ even}\}$ , are linearly independent.*

(2) *Let*

$$\iota: V := \text{span}\{[\sigma_{ij}] \mid i > j \geq 3, i, j \text{ odd}\} \rightarrow H^0(\mathbf{GC})$$

*be the inclusion. Then the homology classes  $\{\iota^*[t_{pq}] \mid p > q > 0, q \text{ even}\}$ , are still linearly independent.*

(3) *Let  $p, q, i, j$  be integers such that  $p > q > 0$ ,  $q$  is even,  $i > j \geq 3$  and  $i$  and  $j$  are odd. Then*

$$\langle t_{pq}, \sigma_{ij} \rangle = \begin{cases} \pm z_{i-p-1}(p-q, j-1) & \text{if } i+j = 2p+q, \text{ and } i > p \geq j \\ 0 & \text{otherwise} \end{cases}$$

*where*

$$(11) \quad z_k(\alpha, \beta) = \binom{k}{\frac{k+\alpha-\beta}{2}} - \binom{k}{\frac{k+\alpha+\beta}{2}}$$

*is the number of zero avoiding  $k$ -step walks from  $\alpha$  to  $\beta$  (see the remark below).*

(4) *The  $t_{pq} \in \mathbf{GC}^*$  vanish on all triple (or higher) commutators in  $\mathbf{GC}$ , i.e.*

$$\langle t_{pq}, [[x, y], z] \rangle = 0$$

*for all  $x, y, z \in \mathbf{GC}$  and for all integers  $p, q$  such that  $p > q > 0, q \text{ even}$ .*

**Remark.** Let  $\alpha, \beta$  be positive integers, and let  $k$  be an integer such that  $k \geq |\alpha - \beta|$  and  $k \equiv \alpha - \beta \pmod{2}$ . A  $k$ -step walk from  $\alpha$  to  $\beta$  is a  $k$ -tuple  $(c_1, \dots, c_k)$  with  $c_1, \dots, c_k \in \{\pm 1\}$  such that  $\beta = \alpha + \sum_{i=1}^k c_i$ . The walk is called zero-avoiding if the partial sums  $\alpha + \sum_{i=1}^j c_i$  are all positive for  $j = 1, \dots, k$ .

<sup>11</sup>Here again we abuse earlier notation and understand  $\sigma_{ij}$  as cocycles in the graph complex, rather than elements of  $\mathbf{grt}$ .

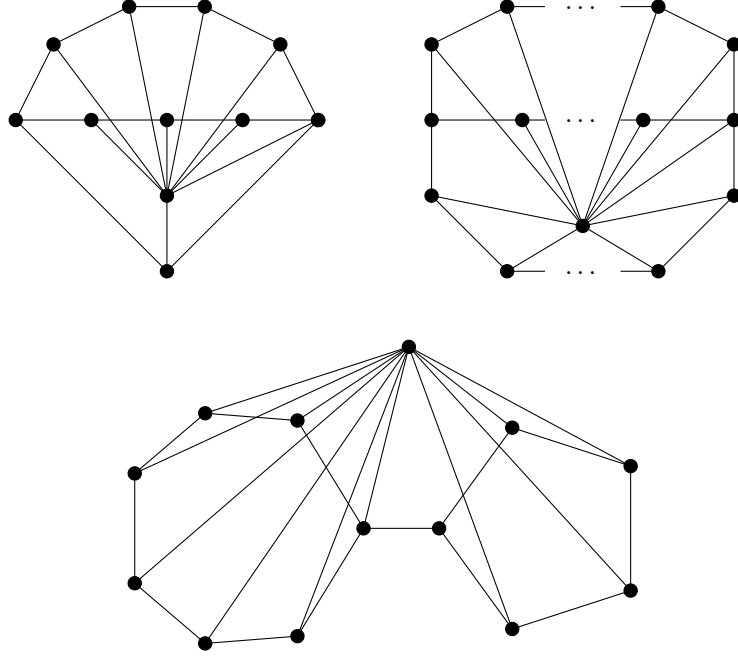


FIGURE 19. The top left graph is the graph  $\Theta(5, 4, 2)$ . On the top right the general scheme for generating the graphs  $\Theta(p, q, r)$  is depicted. The “upper arc” should consist of  $p$  edges, the middle arc of  $q$  and the bottom arc of  $r$  edges. Concretely, it means that the upper “...” represent  $p - 5$  vertices, the middle “...”  $q - 3$  and the lower “...”  $r - 5$  vertices. The bottom graph in this Figure is the graph  $G(7, 5)$ . We have not drawn the general scheme for the  $G(p, q)$  as it should be clear.

*Proof of Proposition 18.* Consider statement 2 of the proposition above. It is clear that the space  $V$  therein decomposes into a direct sum of spaces  $V_N$ . Concretely,  $V_N$  is the subspace of (co)homology classes representable by linear combinations of graphs with  $N + 1$  vertices. On the dual side, the  $t_{pq}$  are linear combinations of graphs with  $2p + q + 1$  vertices. Since the pairing (of course) respects the number of vertices, the restrictions of  $\iota^*[t_{pq}]$  to  $V_N$ , with  $2p + q = N$ , are still linearly independent. A small calculation shows that there are  $\lceil \frac{1}{6}N - 1 \rceil$  allowed combinations of  $p, q$  (i.e.,  $p > q > 0$  and  $2p + q = N$ ), and hence  $V_N$  must have at least that dimension.  $\square$

**Remark.** The Deligne-Drinfeld conjecture asserts that  $\dim V_N = \lfloor \frac{1}{4}N - 1 \rfloor$ . Hence asymptotically as  $N \rightarrow \infty$ , one sees that the homology classes  $[t_{pq}]$  account for  $\frac{2}{3}$  of the expected homology. I tried for some time to find closed formulas for the missing third as well, but without success. Computer experiments indicate that formulas for the missing third, at least if expressed using graphs, are significantly more complicated than the formulas for the  $t_{pq}$ . This is very mysterious to me. What distinguishes the elements of  $\mathbf{grt}^*$  for which there are relative simple formulas from those for which there are none? Is there some underlying reason, or some other way to characterize those elements?

Let us now define the  $t_{pq}$  and give a proof of the above proposition. We will need two different kinds of graphs, denoted  $\Theta(p, q, r)$ , depending on 3 positive integers

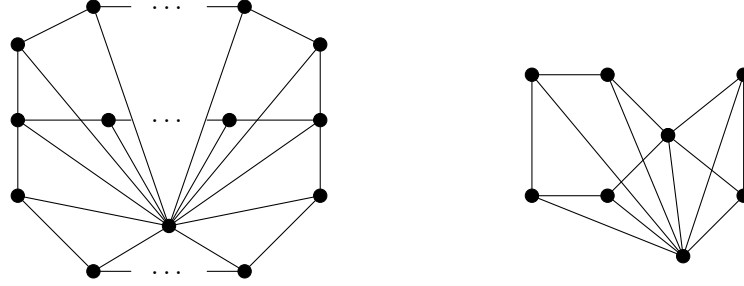


FIGURE 20. On the left the graph  $\hat{\Theta}(p, q, r)$  is depicted. Note that the difference to  $\Theta(p, q, r)$  is merely one edge. However, this edge creates a new symmetry by reflection around a vertical line. Here we allow also that one of the parameters is 0. For example, the graph  $\hat{\Theta}(5, 3, 0)$  is shown on the right.

$p, q, r$  and  $G(p, q)$ , depending on two positive integers  $p, q$ . Instead of giving a formal definition, we refer to Figure 19, from which it should be clear. We will consider  $\Theta(p, q, r)$  and  $G(p, q)$  as elements of  $\mathbf{GC}^*$ . Then the following relations hold by symmetry:

$$\begin{aligned}\Theta(p, q, r) &= \Theta(q, r, p) = -\Theta(q, p, r) \\ G(p, q) &= -(-1)^p G(p, q) = -(-1)^q G(p, q)\end{aligned}$$

In particular, for example  $\Theta(p, p, r) = 0$  for all  $p, r$  and  $G(p, q) = 0$  if  $p$  or  $q$  is even. Now we are ready to write down the formula for  $t_{pq}$ .

$$\begin{aligned}t_{pq} &= \sum_{\substack{0 < i \leq p \\ 0 < j \leq q \\ i > j}} (-1)^{p+q-i-j} z_{p+q-i-j}(p-q, i-j) \Theta(2p+q+1-i-j, i, j) \\ &\quad + \sum_{0 < i \leq p} (-1)^{p+q-i} z_{p+q-i-1}(p-q, i-1) G(2p+q-i, i)\end{aligned}$$

where the  $z_{\dots}(\dots)$  denotes again the number of zero avoiding walks as in equation (11). Note that by the symmetries of  $G(p, q)$ , the terms in the second sum for even  $i$  are all zero. We merely keep them for simplicity of notation,

To see that the  $t_{pq}$  thus defined are actually closed, we need to compute the differentials in  $\mathbf{GC}^*$  of  $\Theta(p, q, r)$  and  $G(p, q)$ . If one defines the signs correctly, one finds that

$$\begin{aligned}\delta\Theta(p, q, r) &= \hat{\Theta}(p-1, q, r) + \hat{\Theta}(p, q-1, r) + \hat{\Theta}(p, q, r-1) \\ \delta G(p, q) &= \hat{\Theta}(p, q, 0)\end{aligned}$$

where the graphs  $\hat{\Theta}(p, q, r)$  are defined graphically in Figure 20. They satisfy the following symmetry properties.

$$\begin{aligned}\hat{\Theta}(p, q, r) &= \hat{\Theta}(q, r, p) = -\hat{\Theta}(q, p, r) \\ \hat{\Theta}(p, q, r) &= (-1)^{p+q+r} \hat{\Theta}(p, q, r) \\ \hat{\Theta}(p, q, 0) &= -(-1)^p \hat{\Theta}(p, q, 0)\end{aligned}$$

Now compute:

$$\begin{aligned} \delta t_{pq} = & \sum_{\substack{0 < i \leq p \\ 0 < j \leq q \\ i > j}} (-1)^{p+q-i-j} z_{p+q-i-j}(p-q, i-j) \\ & \left( \hat{\Theta}(2p+q-i-j, i, j) + \hat{\Theta}(2p+q+1-i-j, i-1, j) + \hat{\Theta}(2p+q+1-i-j, i, j-1) \right) \\ & + \sum_{0 < i \leq p} (-1)^{p+q-i} z_{p+q-i-1}(p-q, i-1) \hat{\Theta}(2p+q-i, i, 0). \end{aligned}$$

Let us collect the coefficient in front of  $\hat{\Theta}(2p+q-i-j, i, j)$  (for  $2p+q-i-j > i > j > 0$ ):

$$\begin{aligned} & (-1)^{p+q-i-j} z_{p+q-i-j}(p-q, i-j) + \delta_{i < p} (-1)^{p+q-i-1-j} z_{p+q-i-1-j}(p-q, i+1-j) \\ & + \delta_{j < q} \delta_{j+1 < i} (-1)^{p+q-i-j-1} z_{p+q-i-j-1}(p-q, i-j-1) \\ & = \pm (z_{p+q-i-j}(p-q, i-j) - \delta_{i < p} z_{p+q-i-1-j}(p-q, i+1-j) \\ & \quad - \delta_{j < q} \delta_{j+1 < i} z_{p+q-i-j-1}(p-q, i-j-1)) \end{aligned}$$

Here  $\delta_{(\dots)}$  is defined to be 1 if the condition  $(\dots)$  is true and 0 otherwise. The term on the right hand side is zero by the following equation for zero-avoiding walks.<sup>12</sup>

$$z_{k+1}(\alpha, \beta) = \delta_{k \geq \beta+1-\alpha} z_{k+1}(\alpha, \beta+1) + \delta_{\beta > 1} \delta_{k \geq \alpha+1-\beta} z_{k+1}(\alpha, \beta-1).$$

We have not yet considered here the  $\hat{\Theta}(2p+q-i, i, 0)$ , i.e., those  $\hat{\Theta}$ 's whose last argument is 0. Those terms are contributed from the second sum, and from the first if  $j = 1$ . The total coefficient is

$$(-1)^{p+q-i-1} z_{p+q-i-1}(p-q, i-1) + (-1)^{p+q-i} z_{p+q-i-1}(p-q, i-1) = 0.$$

Hence we have seen that indeed the  $t_{pq}$  are cocycles.

*Proof of Proposition 19.* The space  $\mathbf{GC}^*$  carries a dg Lie coalgebra structure dual to the dg Lie algebra structure on  $\mathbf{GC}$ . Concretely, the cobracket of a graph  $\Gamma$  is computed by identifying complete subgraphs all of whose vertices are at least trivalent and contracting them:

$$\Delta \Gamma = \sum_{\Gamma' \subset \Gamma} \Gamma / \Gamma' \wedge \Gamma' \cdot (\text{prefactor depending on conventions}).$$

Here  $\Gamma / \Gamma'$  means the graph obtained from  $\Gamma$  by contracting  $\Gamma'$  to one vertex. For example, one can easily see that the cobracket of any wheel graph (see Figure 13) is zero. Similarly, the cobracket of the graphs  $\Theta(p, q, r)$  is zero. The cobracket of the graphs  $G(p, q)$  on the other hand is a product of a wheel graph with  $p$  spokes and one with  $q$  spokes. Hence the double cobracket of any  $t_{pq}$  is zero, which is the last assertion of the proposition. Also, knowing the cobracket of  $t_{pq}$  one computes (using Sweedler notation)

$$\langle t_{pq}, \sigma_{ij} \rangle = \langle t_{pq}, [\sigma_i, \sigma_j] \rangle = \sum \langle t'_{pq}, \sigma_i \rangle \langle t''_{pq}, \sigma_j \rangle = (\text{coefficient of } G(i, j))$$

Hence one obtains assertion 3 of the proposition.

Finally we want to prove assertion 2 from which assertion 1 obviously follows. Without loss of generality we fix the number of vertices to be  $N + 1$ . So assume

<sup>12</sup>The equation is obtained by splitting each walk with  $k + 1$  steps into one with  $k$  steps, plus a last step which can either go up (second term) or down (first term). The  $\delta_{(\dots)}$  in the equation may be omitted if one defines  $z_k(\alpha, \beta) = 0$  for “bad” values of the parameters.

that there is some linear combination  $t$  of the  $t_{pq}$  vanishing on all  $\sigma_{ij}$ . This means that  $t$  has the form

$$t = \sum_{\substack{i>j>1 \\ N>2i+j}} (-1)^{i+j} c_{ij} \Theta(N+1-i-j, i, j)$$

for some yet unknown constants  $c_{ij}$ . Note that there are no  $G(p, q)$ -terms by the condition that  $t$  vanishes on all  $\sigma_{ij}$  and the first part of this proof. We also know that  $t$  is a cycle, i.e.,  $\delta t = 0$ . In terms of the  $c_{ij}$  this amounts to the following system of equations

$$\begin{aligned} c_{i1} &= 0 & \text{for } i \text{ odd} \\ c_{ij} &= \delta_{N>2i+j+1} c_{i+1,j} + \delta_{i>j+1} c_{i,j+1}. \end{aligned}$$

The derivation of these equations proceeds as in the calculation of  $\delta t_{pq} = 0$  above. These equations can be solved by an induction on  $k = i+j$ . The result is that  $c_{ij} = 0$  for all  $i, j$ . This is not obvious, but a standard exercise, so we omit the calculation. Hence we know that  $t = 0$ . On the other hand,  $t$  was a linear combination of  $t_{pq}$ 's, with fixed  $N = 2p + q$ . But the graph  $\Theta(p+1, p, q)$  occurs only in  $t_{pq}$  and hence the linear combination must be trivial.  $\square$

**Remark.** It also follows from Proposition 19 that  $[\sigma_{ij}] \neq 0$  for all odd  $i \neq j$ . This is a well-known result.

#### APPENDIX A. THE DEFORMATION COMPLEX OF $n$ -ALGEBRAS

**A.1. A filtration on  $e_n$  and  $e_n^\vee$ .** The operad  $e_n$  is the operad governing  $n$ -algebras. An  $n$ -algebra is a (graded) vector space  $V$  with binary operations  $\wedge$  of degree 0 and  $[\cdot, \cdot]$  of degree  $1 - n$  satisfying the following relations:

- (1)  $(V, \wedge)$  is a graded commutative algebra.
- (2)  $(V[n-1], [\cdot, \cdot])$  is a graded Lie algebra.
- (3) For all (homogeneous)  $v \in V$ , the unary operation  $[v, \cdot]$  is a derivation of degree  $|v| + 1 - n$  on  $(V, \wedge)$ .

Elements of  $e_n(N)$  can be written as linear combinations of expressions of the form

$$(12) \quad L_1(X_1, \dots, X_N) \wedge \dots \wedge L_k(X_1, \dots, X_N)$$

where  $X_1, \dots, X_N$  are formal variables,  $L_j$  are Lie words and each  $X_i$  occurs exactly once in the expression. The action of the symmetric group  $S_N$  on  $e_n(N)$  is given by permuting the labels on the  $X_1, \dots, X_N$ . From the length of the individual Lie words one can derive various filtrations on  $e_n$ . We will be interested in the filtration coming from the number  $k_1$  of Lie words of length 1. For example, for the expression

$$X_1 \wedge [X_4, X_3] \wedge X_2$$

the number of Lie words of length 1 is  $k_1 = 2$ .

The cooperad  $e_n^\vee$  is the Koszul dual cooperad to  $e_n$ . One can show (see [1]) that  $e_n^\vee = e_n^*[n]$ . Hence from the filtration by  $k_1$  on  $e_n$  one obtains a filtration on  $e_n^\vee$ .

**A.2.  $hoe_n$  and a filtration on the deformation complex.** We define  $hoe_n = \Omega(e_n^\vee)$  as the operadic cobar construction of the cooperad  $e_n^\vee$ . Because  $e_n$  is Koszul (see [10]), there is a (canonical) quasi-isomorphism

$$hoe_n \rightarrow e_n.$$

The deformation complex of  $e_n$  we define as

$$\mathrm{Def}(e_n) = \mathrm{Def}(hoe_n \xrightarrow{id} hoe_n)[1].$$

From the map  $hoe_n \rightarrow e_n$  there is a canonical quasi-isomorphism

$$\mathrm{Def}(e_n) \rightarrow \mathrm{Def}(hoe_n \rightarrow e_n)[1].$$

It is the latter space that we care about in this section. As a vector space, it is a direct product of spaces

$$\mathrm{Hom}_{S_N}(e_n^\vee(N), e_n).$$

For any operad  $\mathcal{P}$  and cooperad  $\mathcal{Q}$ , the product of the spaces

$$\mathrm{Hom}_{S_N}(\mathcal{Q}(N), \mathcal{P}(N))$$

carries a (graded) Lie algebra structure (see [17]), such that the Maurer-Cartan elements correspond to operad maps  $\Omega(\mathcal{Q}) \rightarrow \mathcal{P}$ . In our case ( $\mathcal{P} = e_n$ ,  $\mathcal{Q} = e_n^\vee$ ) the Maurer-Cartan element is the map

$$f = f_\wedge + f_{[\cdot, \cdot]} \in \mathrm{Hom}_{S_2}(e_n^\vee(2), e_n(2))$$

that sends the cocommutative coproduct in  $e_n^\vee(2)$  to the Lie bracket in  $e_2(2)$  and the Lie coproduct in  $e_n^\vee(2)$  to the commutative product in  $e_n(2)$ . The differential on  $\mathrm{Der}(hoe_n \rightarrow e_n)$  is the bracket with  $f$ :

$$d = [f, \cdot] = [f_\wedge, \cdot] = [f_{[\cdot, \cdot]}, \cdot] =: d_1 + d_2.$$

The filtration by  $k_1$  on  $e_2^\vee$  from the previous subsection gives a filtration<sup>13</sup> on  $\mathrm{Def}(hoe_n \rightarrow e_n)$ . The component  $d_1$  can not increase  $k_1$ , while  $d_2$  can increase  $k_1$  by at most one.

**A.3. A more concrete description.** Up to permutations of labels, any element of  $e_n(N)$  can be written as a sum of expressions of the form

$$X_1 \wedge X_2 \wedge \cdots \wedge X_{k_1} \wedge L_1(X_1, \dots, X_N) \wedge \cdots \wedge L_k(X_1, \dots, X_N)$$

where all Lie words  $L_1, \dots, L_n$  have length at least 2. Hence one can decompose

$$e_n(N) = \bigoplus_{k_1=0}^N \mathrm{Ind}_{S_{k_1} \times S_{N-k_1}}^{S_N} (sgn_{k_1}^n \otimes e_n(N-k_1)')$$

where  $e_n(N-k_1)' \subset e_n(N-k_1)'$  is the subspace generated by expressions as above with all Lie words of length at least 2. A similar splitting holds for the  $e_n^\vee$ , and allows us to write

$$\begin{aligned} \mathrm{Hom}_{S_N}(e_n^\vee(N), e_n)_{k_1} &\cong \mathrm{Hom}_{S_{k_1} \times S_{N-k_1}}(sgn_{k_1}^n \otimes e_n^\vee(N-k_1)', e_n(N)) \cong \\ &\cong \mathrm{Hom}_{S_{N-k_1}}(e_n^\vee(N-k_1)', \mathrm{Hom}_{S_{k_1}}(sgn_{k_1}^n, e_n(N))). \end{aligned}$$

where  $\mathrm{Hom}_{S_N}(e_n^\vee(N), e_n)_{k_1} \subset \mathrm{Hom}_{S_N}(e_n^\vee(N), e_n)$  is the subspace of fixed  $k_1$ . The differential  $d$  on  $\mathrm{Der}(hoe_n \rightarrow e_n)$  has some part, say  $d_+$ , that increases  $k_1$  by 1 and some part that leaves  $k_1$  constant or decreases it. We will need an explicit formula for  $d_+$ . Elements of

$$\mathrm{Hom}_{S_{k_1}}(sgn_{k_1}^n, e_n(N)) \cong (sgn_{k_1}^n \otimes e_n(N))^{S_{k_1}}$$

are spanned by expressions  $P(X_1, \dots, X_{k_1}, A_1, \dots, A_m)$ , where  $m = N - k_1$  and  $P$  is a linear combination of expressions as in (12) that is invariant under permutations

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<sup>13</sup>By a filtration on a complex, we mean a filtration on the underlying vector space. We do not require it to be compatible with the differential.



(with signs) of  $X_1, \dots, X_{k_1}$ . The formula for the differential  $d_+$  is:

$$\begin{aligned}
& \pm (d_+ P)(X_1, \dots, X_{k_1+1}, A_1, \dots, A_m) = \\
& = \sum_{i=1}^{k_1+1} (-1)^{n(i+1)} \left[ X_i, P(X_1, \dots, \hat{X}_i, \dots, X_{k_1+1}, A_1, \dots, A_m) \right] \\
& \quad - \sum_{1 \leq i < j \leq k_1+1} (\pm)(-1)^{n(i+j+1)} P([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k_1+1}, A_1, \dots, A_m) \\
& \quad - \sum_{i=1}^{k_1+1} \sum_{j=1}^m (\pm)(-1)^{n(i+1)} P(X_1, \dots, \hat{X}_i, \dots, X_{k_1+1}, A_1, \dots, [X_i, A_j], \dots, A_m) \\
& = \sum_{1 \leq i < j \leq k_1+1} (\pm)(-1)^{n(i+j+1)} P([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k_1+1}, A_1, \dots, A_m).
\end{aligned}$$

Here the signs  $(\pm)$  occur only in the case of even  $n$  and are a bit tricky, owed to the oddness of the Lie bracket.<sup>14</sup> For example, in the third line, one should write down a term of  $P$ , and then compute the number of brackets to the left of the first argument. This gives the sign. The last equality follows since  $[X_i, \cdot]$  is a derivation with respect to the product and the Lie bracket.

**Remark.** In particular, note that  $d_+ = 0$  on the subspace with  $k_1 = 0$ .

**A.4. The cohomology is concentrated in degree  $k_1 = 0$ .** Let  $\Xi = \text{Der}(hoe_n \rightarrow e_n)_{k_1=0}$  be the subspace of  $\text{Der}(hoe_n \rightarrow e_n)$  corresponding to  $k_1 = 0$ . The main result of this section is the following.

**Proposition 20.**  $\Xi \subset \text{Der}(hoe_n \rightarrow e_n)$  is a subcomplex. The inclusion is a quasi-isomorphism.

The first statement follows directly from the Remark in the previous subsection. For the second statement, let us compute the spectral sequence associated to the filtration with  $k_1$ . It is sufficient to show that the second term in that sequence is  $\Xi$ . The latter statement in turn is shown if we can prove that the complexes

$$C_m := \left( \bigoplus_{k_1 \geq 0} (sgn_{k_1}^n \otimes e_n(m + k_1))^{S_{k_1}}, d_+ \right)$$

have cohomology  $\Xi$ . Note that clearly  $Lie_n(m + k_1) \subset e_n(m + k_1)$ . One can make a first reduction on the problem.

**Lemma 6.** *If the complexes*

$$L_m := \left( \bigoplus_{k_1 \geq 1} (sgn_{k_1}^n \otimes Lie_n(m + k_1))^{S_{k_1}}, d_+ \right)$$

*are acyclic, then the complexes  $C_m$  have cohomology  $\Xi$ .*

*Proof.* Consider a complex  $\tilde{C}_m$  that is (like  $C_m$ ) spanned by products

$$P(X_1, \dots, X_{k_1}, A_1, \dots, A_m)$$

of Lie words, but with the  $A_1, \dots, A_m$  allowed to occur with repetitions. It splits as a sum of subcomplexes according to the number of occurrences of  $A_1, A_2$  etc.  $C_m \subset \tilde{C}_m$  is simply the subcomplex in which each  $A_1, A_2, \dots$  occurs exactly once. It is sufficient to prove that the cohomology of  $\tilde{C}_m$  is precisely its  $k_1 = 0$ -part. However,  $\tilde{C}_m$  is a cofree cocommutative coalgebra, the coproduct being the deconcatenation

<sup>14</sup>Strictly speaking, our notation here is almost wrong since different summands of  $P$  may pick up different signs.

of Lie words. The cogenerators are those expressions  $P(X_1, \dots, X_{k_1}, A_1, \dots, A_m)$  containing only a single Lie word. If  $L_m$  is acyclic, the cohomology of the space of generators is precisely its  $k_1 = 0$ -part.  $\square$

Hence we are left with showing that  $L_m$  is acyclic. For  $L_0$  this is easy to see ( $L_0$  is only two-dimensional by the Jacobi identity). Now suppose that  $m \geq 1$ . Then the space of Lie words in  $X_1, \dots, X_{k_1}, A_1, \dots, A_m$ , each symbol occuring exactly once, can be identified with the space of ordinary (associative) words in symbols  $X_1, \dots, X_{k_1}, A_2, \dots, A_m$ , each occuring exactly once. For example (for  $m = 3$ ,  $k_1 = 2$ )

$$[X_1, [A_3, [X_2, [A_2, A_1]]]] \leftrightarrow X_1 A_3 X_2 A_2.$$

A basis for the (anti-)symmetric part in the  $X_j$  can given by symbols like

$$X A_3 X A_2. \leftrightarrow X_1 A_3 X_2 A_2 \pm X_2 A_3 X_1 A_2.$$

In this basis, the differential is given by “doubling”  $X$ ’s, i.e.,

$$d_+(X A_3 X A_2) = X X A_3 X A_2 \pm X A_3 X X A_2.$$

It is well known that this complex is acyclic. An explicit homotopy is “contracting” pairs of  $X$ ’s. Hence the proposition is proven.  $\square$

#### APPENDIX B. $\text{Def}(hoe_n \rightarrow \text{Graphs}_n)$ AND $\text{Def}(hoe_n \rightarrow \text{Gra}_n)$

In this section we give a graphical description of the complexes  $\text{Def}(hoe_n \rightarrow \text{Graphs}_n)$  and its quotient  $\text{Def}(hoe_n \rightarrow \text{Gra}_n)$ . Let us start with the former. We have

$$\text{Def}(hoe_n \rightarrow \text{Graphs}_n) \cong \prod_N \text{Hom}_{S_N}(e_n^\vee(N), \text{Graphs}_n(N)).$$

As in Appendix A, this space is a graded Lie algebra and the differential is given by the bracket with a Maurer-Cartan element. Here the Maurer-Cartan element is the image  $\tilde{f} = \tilde{f}_\wedge + \tilde{f}_{[\cdot, \cdot]}$  under  $\text{Def}(hoe_n \rightarrow e_n) \rightarrow \text{Def}(hoe_n \rightarrow \text{Graphs}_n)$  of the map  $f = f_\wedge + f_{[\cdot, \cdot]}$  of Appendix A. More precisely,  $\tilde{f}$  maps the coproduct in  $e_n^\vee(2)$  to the graph in  $\text{Graphs}_n(2)$  with two vertices and one edge connecting them, and maps the cobracket to the “empty” graph with two vertices and no edge.

For later proofs, it is helpful to have a description of  $\text{Der}(hoe_n \rightarrow \text{Graphs}_n)$  as a graph complex. Elements of  $e_n^\vee(N)$  can be written as linear combinations of expressions like

$$(324) \wedge (51) \wedge (6)$$

(for  $n = 6$ ), where the order of the brackets is arbitrary (up to sign) and the order of numbers within the brackets is defined only modulo shuffles. One can decompose any element of  $e_n^\vee(N)$  according to the number of brackets with  $j$  numbers inside. Let

$$e_n^\vee(N)_{k_1, k_2, \dots}$$

Be the space of elements of  $e_n^\vee(N)$  with  $k_1$  brackets of length 1,  $k_2$  brackets of length 2 etc. Here  $N = \sum_j j k_j$ .

**Remark.** One may represent  $e_n^\vee$  using graphs. Then  $e_n^\vee(n)_{k_1, k_2, \dots}$  is the space of graphs with  $k_1$  isolated vertices,  $k_2$  connected components of size 2,  $k_3$  connected components of size 3 etc.

**Lemma 7.** *The space  $\text{Hom}_{S_N}(e_n^\vee(N)_{k_1, k_2, \dots}, \text{Graphs}_n(N))$  is isomorphic to the space of (linear combinations of) graphs in  $\text{Graphs}_n(N)$  that have the following symmetry properties.*

$$\begin{aligned}
& \text{Diagram 1} \mapsto \sum \pm \text{Diagram 2} \quad \text{Diagram 3} \mapsto \sum \pm \text{Diagram 4} \\
& ( \text{oooooo} ) \mapsto \sum \pm \left( \begin{array}{c} \text{oooo} \\ \text{ooo} \end{array} \right) \\
& ( \text{ooooo} ) \mapsto \sum \pm ( \text{oooooo} )
\end{aligned}$$

FIGURE 21. The various parts of the differential on the complex  $\text{Def}(hoe_n \rightarrow \mathbf{Graphs}_n)$ . For a detailed description, see the text below.

- (1) Consider the input vertices as organized into clusters, with  $k_1$  clusters of 1 vertices,  $k_2$  clusters of 2 vertices, etc. Then the (linear combination of) graphs  $\Gamma \in \mathbf{Graphs}_n(N)$  must be invariant under interchange (with sign) of clusters of the same size. The sign is  $(-1)^{j+n+1}$  for the interchange of clusters of size  $j$ .
- (2) We fix a linear ordering on the vertices of each cluster. Then  $\Gamma$  must “vanish on shuffles” in any cluster. This means the following. Fix some cluster of length  $j$ , and  $j_1, j_2 \geq 1$  s.t.  $j = j_1 + j_2$ . Let  $USh_{j_1, j_2}$  be the set of  $(j_1, j_2)$ -unshuffle permutations, which we consider acting on  $\mathbf{Graphs}(n)$  by permuting the vertices in the cluster under consideration. Then we require

$$\sum_{\sigma \in USh_{j_1, j_2}} \pm \sigma \cdot \Gamma = 0.$$

The complex  $\text{Def}(hoe_n \rightarrow \mathbf{Gra}_n)$  has a similar interpretation in terms of graphs. Indeed, it is obtained from  $\text{Def}(hoe_n \rightarrow \mathbf{Graphs}_n)$  by sending all graphs with internal vertices to zero. The complexes  $\text{Def}(hoe_n \rightarrow \mathbf{Gra}_n^\odot)$  and  $\text{Def}(hoe_n \rightarrow \mathbf{Graphs}_n^\odot)$  can similarly be given a graphical description, by just allowing tadpoles in the graphs.

**Remark.** The degree of a graph can be computed as  $n \cdot (\# \text{vertices} - 1) - (n - 1) \cdot (\# \text{edges})$ , when we count a cluster with  $k$  vertices as  $k$  vertices (of course) and  $k - 1$  edges.

**B.1. A Graphical description of the differential.** The differential on  $\text{Def}(hoe_n \rightarrow \mathbf{Graphs}_n)$  (described in graphical terms) has four parts:

- (1) The first part splits an internal vertex into two internal vertices. It comes from the differential on  $\mathbf{Graphs}_n$ .
- (2) The second part splits an external vertex into an external and an internal vertex. It also comes from the differential on  $\mathbf{Graphs}_n$ .
- (3) The third part,  $d_L$ , splits a cluster of external vertices into two clusters, by splitting one external vertex in that cluster. It creates an edge between the two vertices that the original vertex was split up into. This part of the differential comes from the bracket (on  $\text{Def}(hoe_n \rightarrow \mathbf{Graphs}_n)$ ) with  $\tilde{f}_{[\cdot, \cdot]}$ . (For details, see Figure 21.)
- (4) The fourth part,  $d_H$  (H for “Harrison”), also creates an external vertex, but does not split the cluster and does not introduce a new edge. Hence it maps a cluster of length  $j$  to one of length  $j + 1$ . (For details, see Figure 21.)

In the complex  $\text{Def}(hoe_n \rightarrow \mathbf{Gra}_n)$  the first two terms are absent.

**B.2. The cohomology of  $\text{Def}(hoe_n \rightarrow \mathbf{Gra}_n)$  and  $\text{Def}(hoe_n \rightarrow \mathbf{Gra}_n^\odot)$ .** The goal of this subsection is to prove the following Proposition.

**Proposition 21.** *The cohomology of  $\text{Def}(\text{hoe}_n \rightarrow \text{Gra}_n^\circ)_{\text{conn}}$  is*

$$H(\text{Def}(\text{hoe}_n \rightarrow \text{Gra}_n^\circ)_{\text{conn}}) \cong \mathbb{R}[-1]$$

*and for even  $n$ , the cohomology of its tadpole free part is*

$$H(\text{Def}(\text{hoe}_n \rightarrow \text{Gra}_n)_{\text{conn}}) \cong \mathbb{R}[-1] \oplus \mathbb{R}[n-2]$$

**Remark.** The cohomology class  $\mathbb{R}[-1]$  corresponds to a relative rescaling of the Lie bracket and product. The class  $\mathbb{R}[n-2]$  encodes a map sending the Lie co-bracket to the Lie bracket. The corresponding graph has two vertices in a cluster and an edge connecting them. In the complex with tadpoles this class is the coboundary of the “divergence” operation, i.e., the graph with one vertex and a tadpole.

**Remark.** For  $n = 2$  this result is the “universal” version of the well-known statement that the Gerstenhaber algebra  $T_{\text{poly}}$  is non-deformable. (The operads  $\text{Gra}_2$  or  $\text{Gra}_2^\circ$  can be seen as universal versions of the operad  $\text{End}(T_{\text{poly}})$ .)

We disregard the connectedness requirement at the beginning. We simply compute all classes and at the end keep only the connected ones.

Let us treat first the case with tadpoles. We use here the graphical interpretation of  $\text{Def}(\text{hoe}_n \rightarrow \text{Gra}_n^\circ)$  and terminology from the previous subsection. Consider the filtration coming from the quantity  $l = \#(\text{vertices}) - \#(\text{clusters})$ . The differential, say  $d$ , increases  $l$  by at most 1. Take the spectral sequence associated to  $l$ . Then the differential on the first term will be given by  $d_H$ , which does not alter  $l$ . As  $d_H$  does not produce new edges, the complex splits into a direct sum of subcomplexes, according to the number of edges. Say we fix one subcomplex  $C_K$  consisting of graphs with  $K$  edges. Let  $\tilde{C}_K$  be the complex obtained by giving each half-edge a unique identifier, say  $a_1, b_1, a_2, b_2, \dots, a_K, b_K$ . Then  $C_K = \tilde{C}_K / G$  where the group  $G := S_2^{\times K} \times S_K$  acts by permutations (possibly with signs) of the labels on the edges. Since the differential commutes with the  $G$ -action, the cohomology of  $C_K$  is

$$H(C_K) = H(\tilde{C}_K) / G.$$

Let us compute the cohomology of  $\tilde{C}_K$ . We can represent each graph by an expression of the form (in this example  $K = 3$ ).

$$(13) \quad (a_3 b_1, , b_3)(a_1 b_2, , a_2, a_4)(b_4)$$

It is to be read as follows: There are three clusters, each corresponding to a pair of brackets  $(\dots)$ . The first cluster contains 3, the second 4, and the third 1 vertex. The first vertex in cluster 1 is connected to the half-edges  $a_3$  and  $b_1$ , the second vertex is not connected to anything, and the third vertex is connected to the half-edge  $b_3$ , etc. The order of the brackets  $(\cdot)$  (i.e., the clusters) and the order of the halfedges for each vertex does not matter up to sign. complex  $\tilde{C}_K$  is spanned by linear combinations of expressions like the one above, that are in addition required to “vanish on shuffles”, i.e., if we take the sum (with signs) over unshuffles of vertices in one cluster, the result vanishes. Let us introduce yet another complex. Let  $\bar{C}_K$  be defined similar to  $\tilde{C}_K$ , except that each of the symbols  $a_1, b_1, \dots, b_K$  is no longer required to occur exactly once in an expression of the form (13). The complex  $\bar{C}_K$  splits into a direct sum of subcomplexes, according to the numbers of symbols  $a_1, b_1, \dots, b_K$  occurring in an expression. The complex  $\tilde{C}_K \subset \bar{C}_K$  is simply the subcomplex in which each symbol occurs exactly once. But  $\tilde{C}_K$  is isomorphic (up to degree shifts) to the symmetric part of a tensor product of Harrison complexes

$$THarr(\mathbb{F}_{\text{coCom}}(a_1, b_1, \dots, b_K)).$$

Here  $\mathbb{F}_{\text{coCom}}(a_1, b_1, \dots, b_K)$  is the free cocommutative coalgebra cogenerated by symbols  $a_1, b_1, \dots, b_K$  and  $Harr(\dots)$  is its cohomological Harrison complex. The

differential  $d_H$  is just the Harrison differential, whence the notation. But the Harrison cohomology of a free coalgebra is zero except in degree one, where it is the space of generators.

Translated in the language of graphs, it means that each cohomology class in  $\mathrm{Der}(hoe_n \rightarrow \mathrm{Gra}_n^\circ)$  can be represented by a graph with all clusters of length 1 (single vertices), and each edge is hit by exactly one edge. Considering only connected graphs, we are left with the graph with 2 vertices connected by one edge. This cohomology class corresponds to a rescaling of the bracket (but not the product) of an  $n$ -algebra. For degree reasons the spectral sequence degenerates at this point and hence the above Proposition is shown.

Next let us consider even  $n$  and the tadpole-free case. The arguments proceed along the same lines, except that now  $\bar{C}_K$  is the symmetric subspace (up to degree shifts) of

$$THarr(\mathbb{F}'_{coCom}(a_1, b_1, \dots, b_K))$$

where  $\mathbb{F}'_{coCom}(a_1, b_1, \dots, b_K)$  is the cocommutative coalgebra cogenerated by symbols  $a_1, b_1, \dots, b_K$ , with co-relations  $a_j b_j \mapsto 1$ . This algebra is a  $K$ -fold coproduct of a cocommutative coalgebra  $\mathbb{F}'_{coCom}(a, b)$ , hence by the Künneth formula, its Harrison cohomology is the  $K$ -fold sum of the cohomology of  $\mathbb{F}'_{coCom}(a, b)$ . The Harrison cohomology of the latter coalgebra is three-dimensional, concretely, it is the Lie algebra with odd generators  $\tilde{a}, \tilde{b}$  and relations  $[\tilde{a}, \tilde{a}] = [\tilde{b}, \tilde{b}] = 0$ . In terms of graphs this implies that there might be additional graph cohomology classes with clusters of size two, the vertices of which are connected by an edge. Since we consider here only connected graphs only one such cluster is allowed, yielding the term  $\mathbb{R}[n-2]$  in the cohomology.  $\square$

**B.3. The first term of the spectral sequence for  $\mathrm{Def}(hoe_n \rightarrow \mathrm{Graphs}_n)_{conn}$  and  $\mathrm{Def}(hoe_n \rightarrow \mathrm{Graphs}_n^\circ)_{conn}$ .** One can set up a spectral sequence for the computation of the cohomology of  $\mathrm{Def}(hoe_n \rightarrow \mathrm{Graphs}_n)_{conn}$  and  $\mathrm{Def}(hoe_n \rightarrow \mathrm{Graphs}_n^\circ)_{conn}$ , such that the first differential is the one increasing the size of the clusters, as we did in the previous subsection. Let us call this differential again  $d_H$ . One can apply (almost) the same reasoning as before to conclude that the cohomology of  $\mathrm{Def}(hoe_n \rightarrow \mathrm{Graphs}_n^\circ)_{conn}$  under  $d_H$  is given by graphs with all clusters of size one, and all external vertices univalent. For even  $n$  the cohomology of  $\mathrm{Def}(hoe_n \rightarrow \mathrm{Graphs}_n)_{conn}$  under  $d_H$  is given by the same graphs, only without tadpoles, plus the graph with two vertices in a cluster, connected by an edge. But the graphs of these shapes span subcomplexes, let us call them  $C^\circ \subset \mathrm{Def}(hoe_n \rightarrow \mathrm{Graphs}_n^\circ)_{conn}$  and  $C \subset \mathrm{Def}(hoe_n \rightarrow \mathrm{Graphs}_n)_{conn}$  ( $n$  even).

**Proposition 22.** *The inclusions  $C^\circ \subset \mathrm{Def}(hoe_n \rightarrow \mathrm{Graphs}_n^\circ)_{conn}$  and  $C \subset \mathrm{Def}(hoe_n \rightarrow \mathrm{Graphs}_n)_{conn}$  ( $n$  even) are quasi-isomorphisms.*

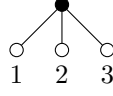
*Proof.* If one has a subcomplex and a convergent spectral sequence reducing to this subcomplex at the first step, the inclusion is a quasi-isomorphism. The author learned this trick from P. Ševera.  $\square$

## APPENDIX C. (RE-)DERIVATION OF FURUSHO'S RESULT

In a remarkable paper [9], H. Furusho showed that the Hexagon equation (4) is a consequence of the Pentagon equation (3), if one requires that  $\psi \in F_{Lie}(x, y)$  (as in those equations) does not contain the term  $[x, y]$ . Rephrasing this result in operadic language, it reads as follows.

**Proposition 23.**

$$H^1(\mathrm{Def}(Ass_\infty \rightarrow t[1])) \cong H^1(\mathrm{Def}(Com_\infty \rightarrow t[1])) \oplus \mathbb{R}[-1].$$

FIGURE 22. The graph corresponding to the  $[t_{12}, t_{23}] \in \mathfrak{t}_3$ .

Here the  $\mathbb{R}[-1]$  corresponds to the cohomology class represented by  $\psi = [x, y]$ .

*Proof.* We will actually show that  $H^1(\text{Def}(Ass_\infty \rightarrow \text{ICG}^\circ[1])) \cong H^1(\text{Def}(Com_\infty \rightarrow \text{ICG}^\circ[1])) \oplus \mathbb{R}[-1]$ . Take a spectral sequence as in the proof of Proposition 21. The difference to the situation there is that we now have to compute the Hochschild instead of the Harrison cohomology of a cofree cocommutative coalgebra. But it is well known that the Hochschild cohomology of such an algebra is the Koszul dual coalgebra, i.e., an (anti-)commutative free algebra. Translated into graphical language, it means that the cohomology is given by graphs, whose external vertices are connected by exactly one edge each, and which are antisymmetric under interchange of external vertices.

Suppose we are given a closed (sum of) graphs  $x$ , with  $\leq n$  external vertices. We split it according to the number of external vertices as  $x = x_n + x_{n-1} + \dots$ . Similarly, we split the differential into a part  $d_s$  creating an external vertex, and a part  $\delta$  not creating one. In fact,  $\delta$  is the part of the differential coming from the differential on  $\text{ICG}^\circ$ . The above argument implies that we can assume  $x_n$  to be antisymmetric, and having each external vertex of valence 1.

Claim 1:  $d_s x_n = \delta x_n = 0$ . Indeed,  $d_s x_n = 0$  by degree reasons. Furthermore one checks that  $d_s x_{n-1}$  cannot contain any antisymmetric part. But  $\delta x_n$  is antisymmetric and hence the claim follows. Note that Claim 1 means that  $x_n$  is already a cocycle.

Claim 2:  $x_n$  represents a trivial cohomology class except possibly in the following cases:

- (1)  $n = 3$  and  $x_n$  contains the graph, say  $T_3$ , with one internal vertex, see Figure 22. This graph corresponds to  $[t_{12}, t_{23}] \in \mathfrak{t}_3$ .
- (2)  $n = 1$

From Claim 2, the Proposition follows by a simple induction. To prove the claim, note that  $\text{Def}(Ass_\infty \rightarrow \text{ICG}^\circ[1])$  is a double complex and compute the spectral sequence whose first convergent is the  $\delta$ -cohomology. As we noted before, the cohomology of  $\text{ICG}$  is  $\mathfrak{t}$ , concentrated in degree 0. Hence, by degree reasons,  $x_n$  cannot describe a non-trivial cohomology class if  $n > 3$ . For  $n = 3$ ,  $x_n$  is  $\delta$ -exact, unless it contains a tree part. The only tree with all external vertices of valence 1 is  $T_3$ . Hence, if  $x_n$  doesn't contain a  $T_3$ -part, the same argument as for the case  $n > 3$  works.

For  $n = 2$  we distinguish two cases: Both external vertices are connected to the same internal vertex, or two different ones. In the case “same” one can see that  $x_n = \delta y$  for some antisymmetric  $y$  having degree 1 external vertices. Hence  $d_s y = 0$  and  $x_n$  is exact. In the case “different”,  $x_n = \delta y$  for a  $y$  obtained by contracting one of the external edges. One checks that combinatorially  $d_s y$  cannot contain any trivalent-tree-part: Such a tree must necessarily have (at least) two internal vertices connected to different external vertices. Since one external vertex of every graph in  $y$  has valence one, it means that one of the internal vertices must have two edges connected to the same external vertex, which is a contradiction (or rather, the graph is 0). Hence  $d_s y = \delta z$  for some  $z$  and hence by the same reasoning as in the  $n > 3$ -case, the cohomology class of  $x_n$  vanishes.  $\square$

APPENDIX D. THE ONE VERTEX IRREDUCIBLE PART OF  $\mathbf{GC}_n$  IS  
QUASI-ISOMORPHIC TO  $\mathbf{GC}_n$

Probably everything in this subsection has been shown by M. Kontsevich, but has not been written up. Let  $\mathbf{GC}_n^{1vi} \subset \mathbf{GC}_n$  be the subspace of 1-vertex irreducible graphs, i.e., those graphs that remain connected after deleting one vertex.

**Lemma 8.**  $\mathbf{GC}_{1vi} \subset \mathbf{GC}$  is a sub-dg Lie algebra.

*Proof.* It is clear. □

The following Proposition has been shown by Conant, Gerlits and Vogtman.

**Proposition 24** ([4]).  $\mathbf{GC}_n^{1vi} \hookrightarrow \mathbf{GC}$  is a quasi-isomorphism.

We nevertheless give a different short sketch of proof for completeness, following the idea of Lambrechts and Volic [16].

(*very sketchy*) *Sketch of proof.* We need to show that  $\mathbf{GC}_n/\mathbf{GC}_n^{1vi}$  is acyclic. Any non-1vi graph can be written using the following data: (i) a family of 1-vertex-irreducible graphs (“1vi components”) (ii) a tree (iii) for each vertex in the tree, a subset of vertices in the irreducible components. All vertices in that subset are glued together to form the graph. The differential can be decomposed into two parts: One part that changes one of the 1vi components, and one that does not. One can set up a spectral sequence, such that its first term is the latter part of the differential (leaving invariant the 1vi components).<sup>15</sup> The resulting complex splits into subcomplexes according to the family of 1vi components. Fix one such subcomplex, say  $C$ , and fix one vertex  $v$  in one of the 1vi components, that belongs to the subset associated to vertex  $t$  of the tree. There is a filtration  $C \supset C_1$ , where  $C_1$  is the subspace containing graphs such that the number of vertices in the subset of  $t$  is one and  $t$  has only one incident edge. Take the spectral sequence. Its first term contains a differential mapping

$$C/C_1 \rightarrow C_1$$

which can easily be seen to be an isomorphism. □

APPENDIX E. A NOTE ON THE CONVERGENCE OF SPECTRAL SEQUENCES

In this paper, we use spectral sequences to compute the cohomology of graph complexes at various places, in particular for  $\mathbf{GC}_n$  and  $\mathbf{fGC}_n$ . We claim that these spectral sequences indeed converge to the cohomology.

For  $\mathbf{GC}_n$  this is very easy to see.  $\mathbf{GC}_n$  splits into a direct sum of finite dimensional subcomplexes, for fixed values of the difference

$$\Delta = e - v := \# \text{edges} - \# \text{vertices}.$$

Indeed, since each vertex is at least trivalent, one has  $e \geq \frac{3}{2}v$  and hence  $v \leq 2\Delta$  is bounded within each subcomplex. But there are only finitely many graphs with fixed  $\Delta$  and bounded  $v$ , and hence each subcomplex is finite dimensional.

For  $\mathbf{fGC}_n$  the argument is a bit more subtle. Let  $\mathcal{F}$  be a filtration on  $\mathbf{fGC}_n$ , compatible with the differential. Let  $\mathbf{fGC}'_n$  be the same complex as  $\mathbf{fGC}_n$ , but with the degrees shifted by  $(n - \frac{1}{2})\Delta$ , so that the new degree of a graph is

$$n(v - 1) - (n - 1)e + (n - \frac{1}{2})\Delta = \frac{1}{2}(v + e) - n.$$

It is clear that (i) each grading component of  $\mathbf{fGC}'_n$  is finite dimensional, that (ii) the cohomology of  $\mathbf{fGC}'_n$  is the same as that of  $\mathbf{fGC}_n$  up to degree shifts and that

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<sup>15</sup>Note that the spectral sequence converges because the graph complex splits into finite dimensional subcomplexes.

(iii) one has a filtration  $\mathcal{F}'$  on  $\mathbf{fGC}'_n$ . By (i) the filtration  $\mathcal{F}'$  is bounded and hence the associated spectral sequence converges to the cohomology of  $\mathbf{fGC}'_n$ . But the spectral sequence associated to  $\mathcal{F}$  is the same as that associated to  $\mathcal{F}'$ , up to some degree shifts. Hence it converges to the degree shifted cohomology of  $\mathbf{fGC}'_n$ , which by (ii) is the cohomology of  $\mathbf{fGC}_n$ .

#### APPENDIX F. $\mathbf{t}$ , $\mathbf{grt}$ , $\mathbf{sder}$

Let  $\mathbf{sder}$  be the operad of Lie algebras of special derivations of free Lie algebras (see [1], or [7]). Elements of  $\mathbf{sder}$  can be seen as internal trivalent trees in  $\mathbf{ICG}$ , modulo the Jacobi identity. As noted in [18] there is a spectral sequence coming from the filtration on  $\mathbf{ICG}$  by internal loops, whose first term contains  $\mathbf{sder}$ .

In particular,  $\mathbf{t}$  is a sub-operad of Lie algebras of  $\mathbf{sder}$ . The map  $\mathbf{t} \rightarrow \mathbf{sder}$  sends a generator  $t_{ij}$  to the graph with a single edge between external vertices  $i$  and  $j$ .

**Proposition 25.** *Let  $\phi \in \mathbf{t}_3$  and  $\Gamma$  its image in  $\mathbf{sder}(3)$ . Then  $\phi$  can be recovered from  $\Gamma$  as follows:*

- (1) Forget all graphs in  $\Gamma$  that have more than one edge incident to vertex 2.
- (2) The coefficient  $c$  of the (single possible) graph with no vertex at 2 yields the coefficient of  $t_{13}$  in  $\phi$ .
- (3) Interpret each remaining tree as a Lie tree rooted at 2, corresponding to a Lie expression  $\psi \in F_{Lie}(X, Y)$ .
- (4) Then  $\phi = c \cdot t_{13} + \psi(t_{12}, t_{13})$ .

*Proof.* By induction on the degree. □

#### APPENDIX G. TWISTING OF OPERADS

**G.1. A Foreword.** Any operad describes a certain kind of algebraic object. Often the algebraic object (the representation of the operad) is easier to describe and comprehend than the operad itself. So let us describe first what algebraic situation we want to consider by defining a “twisted operad”.

Suppose we are given some operad  $\mathcal{P}$  together with a  $\mathcal{P}$ -algebra  $A$ . Suppose further that we have a map  $hoLie \rightarrow \mathcal{P}$ . In particular, this means that  $A$  is also a  $hoLie$ -algebra. Sweeping convergence issues under the rug, it makes sense to talk about Maurer-Cartan elements in  $A$ , which are simply Maurer-Cartan (MC) elements in the  $hoLie$ -algebra  $A$ . Fix such an MC element  $m$ . Twisting the  $hoLie$  structure on  $A$  by  $m$ , one can in particular endow  $A$  with a new differential. Furthermore one can construct new operations on  $A$  by inserting  $m$ ’s into the  $\mathcal{P}$ -operations on  $A$ . The twisted operad  $Tw\mathcal{P}$  is defined such that the algebra  $A$  with twisted differential and with this extended set of operations is a  $Tw\mathcal{P}$ -algebra.

The important claim for this paper is that there is an action of the deformation complex  $\mathbf{Def}(hoLie \rightarrow \mathcal{P})$  on the operad  $Tw\mathcal{P}$ . Concretely, it is defined as follows: Suppose that we have a closed degree zero element  $x \in \mathbf{Def}(hoLie \rightarrow \mathcal{P})$ . Such an element gives a  $hoLie$  derivation (i.e., infinitesimal automorphism) of the  $hoLie$ -algebra  $A$ . Having an MC element  $m \in A$ , one can twist this derivation by  $m$ . One obtains in particular an (infinitesimally) different MC element  $m' \in A$  and hence also a new  $Tw\mathcal{P}$ -structure on  $A$ . The action of  $x$  on the operad  $Tw\mathcal{P}$  is defined such that it induces that change of  $Tw\mathcal{P}$ -structure.

**Remark.** It is important that we define  $Tw\mathcal{P}$  such that  $A$  as above with differential twisted by  $m$  is a  $Tw\mathcal{P}$ -algebra.



**G.2. The construction.** Let  $\mathcal{P}$  be any (dg) operad. We will consider it here as a contravariant functor from the category of finite sets (with bijections as morphisms) to the category of dg vector spaces. For a finite set  $S$ , the space  $\mathcal{P}(S)$  can be seen as the space of  $\#S$ -ary operations, with inputs labelled by elements of  $S$ . We will write for short  $\mathcal{P}(n) := \mathcal{P}(\{1, \dots, n\})$ . For some operation  $a \in \mathcal{P}(n)$  and for some symbols  $s_1, \dots, s_n$  we will write

$$a(s_1, \dots, s_n) \in \mathcal{P}(\{s_1, \dots, s_n\})$$

for the image of  $a$  under the map  $\mathcal{P}(f)$ , where  $f : \{s_1, \dots, s_n\} \rightarrow [n]$  is the bijection sending  $s_j \mapsto j$ . Similarly, for  $a \in \mathcal{P}(n)$ ,  $b \in \mathcal{P}(m)$ , and symbols  $s_1, \dots, s_{m+n-1}$  we will write

$$a(s_1, \dots, s_{j-1}, b(s_j, \dots, s_{j+m-1}), s_{j+m}, \dots, s_{m+n-1}) = (a \circ_j b)(s_1, \dots, s_{m+n-1})$$

for the operadic composition.

Let  $hoLie_k := (Lie\{k\})_\infty$  be the minimal resolution of the degree shifted Lie operad. Concretely, the Lie bracket here has degree  $-k$ . An operad map  $hoLie_k \rightarrow \mathcal{P}$  is described by a Maurer-Cartan element  $\mu$  in

$$\mathfrak{g} = \text{Def}(hoLie_k \rightarrow \mathcal{P}) \cong \prod_j (\mathcal{P}(j) \otimes (\mathbb{R}[-k-1])^{\otimes j})^{S_j}[k+1].$$

On the right, the permutation group  $S_j$  acts on  $\mathcal{P}(j)$  as usual and on the tensor product of  $\mathbb{R}[-k-1]$ 's is by permutation, with appropriate signs. Up to degree shift, elements of  $\mathfrak{g}$  are sums of symmetric or antisymmetric elements of  $\mathcal{P}$ .

**Remark.** Fixing an MC  $\mu \in \mathfrak{g}$  fixes the notion of Maurer-Cartan element in any (nilpotent)  $\mathcal{P}$ -algebra  $A$ . Concretely, the latter is just a Maurer-Cartan element in the  $hoLie_k$ -algebra  $A$ .

We next want to define an operad  $Tw\mathcal{P}$  that governs  $\mathcal{P}$ -algebras “twisted by” a Maurer-Cartan element. The underlying functor be defined on a set  $S$  as

$$Tw\mathcal{P}(S) = \prod_{j \geq 0} (\mathcal{P}(S \sqcup \{\bar{1}, \dots, \bar{j}\}) \otimes (\mathbb{R}[k+1])^{\otimes j})^{S_j}$$

where the symmetric group  $S_j$  acts by permutation on the symbols  $\bar{1}, \dots, \bar{j}$  and by appropriate permutation (with signs) on the  $\mathbb{R}[k+1]$ . An element of the  $j$ -th factor in the product should be seen as an operation in  $\mathcal{P}$  invariant under permutations of the last  $j$  slots. For an element  $a$  in the  $j$ -th factor of  $Tw\mathcal{P}(n)$  and some symbol set  $S = \{s_1, \dots, s_{n+j}\}$  the expression

$$a(s_1, \dots, s_n, s_{n+1}, \dots, s_{n+j}) \in \mathcal{P}(S)[j(k+1)]$$

accordingly makes sense. The operadic compositions are defined for homogeneous (wrt. both the grading by  $j$  and the degree) elements  $a \in Tw\mathcal{P}(m)$ ,  $b \in Tw\mathcal{P}(n)$

$$\begin{aligned} & (a \circ_l b)(s_1, \dots, s_{m+n-1}, \bar{1}, \dots, \overline{j_1 + j_2}) = \\ &= \sum_{I \sqcup J = [j_1 + j_2]} \text{sgn}(I, J)^{k+1} (-1)^{|b|j_1(k+1)} a(s_1, \dots, s_{l-1}, b(s_l, \dots, s_{l+n-1}, \bar{J}), \dots, s_{m+n-1}, \bar{I}) \\ & \in (\mathcal{P}(S \sqcup \{\bar{1}, \dots, \overline{j_1 + j_2}\}) \otimes (\mathbb{R}[k+1])^{\otimes (j_1 + j_2)})^{S_{j_1 + j_2}} \end{aligned}$$

where  $\bar{I}$  is shorthand for  $\bar{i}_1, \bar{i}_2, \dots$  with  $i_1 < i_2 < \dots$  being the members of the set  $I$ , and similarly for  $\bar{J}$ . The sign is the sign of the shuffle permutation bringing  $i_1, \dots, j_1, \dots$  in the correct order.

**Remark.** An element of  $Tw\mathcal{P}$  should be thought of as an operation in  $\mathcal{P}$ , with Maurer-Cartan elements inserted into some of its slots.

Next we want to define a differential on  $Tw\mathcal{P}$ . First, denote by  $\widetilde{Tw}\mathcal{P}$  the operad  $Tw\mathcal{P}$  as defined so far, with the differential solely the one coming from the differential on  $\mathcal{P}$ . On  $\widetilde{Tw}\mathcal{P}$  we have a right action of the (dg) Lie algebra  $\mathfrak{g}$ . Concretely, for homogeneous  $x \in \mathfrak{g}$ ,  $a \in \widetilde{Tw}\mathcal{P}(n)$  we have

$$(a \cdot x)(1, \dots, n, \bar{1}, \dots, \overline{j_1 + j_2}) = \sum_{I \sqcup J = [j_1 + j_2]} sgn(I, J)^{k+1} a(1, \dots, n, \bar{I}, x(\bar{J})).$$

**Lemma 9.** *This formula describes a right action by operadic derivations.*

*Proof.* For homogeneous elements we compute

$$\begin{aligned} & ((a \circ_l b) \cdot x)(s_1, \dots, s_{m+n-1}, \bar{1}, \dots, \overline{j_1 + j_2 + j_3}) \\ &= \sum_{I \sqcup J \sqcup K = [j_1 + j_2 + j_3]} sgn(I \cup J, K)^{k+1} sgn(I, J)^{k+1} (-1)^{|b|j_1(k+1)} \\ & \quad a(s_1, \dots, s_{l-1}, b(s_l, \dots, s_{l+n-1}, \bar{J}), \dots, s_{m+n-1}, \bar{I}, x(\bar{K})) \\ & \quad + sgn(I \cup J, K)^{k+1} sgn(I, J)^{k+1} (-1)^{(|b|+|x|)j_1(k+1)} \\ & \quad a(s_1, \dots, s_{l-1}, b(s_l, \dots, s_{l+n-1}, \bar{J}, x(\bar{K})), \dots, s_{m+n-1}, \bar{I}) \\ &= ((-1)^{|b||x|} (a \cdot x) \circ_l b + a \circ_l (b \cdot x))(s_1, \dots, s_{m+n-1}, \bar{1}, \dots, \overline{j_1 + j_2 + j_3}) \end{aligned}$$

For the last equality we used that  $sgn(I \cup J, K)sgn(I, J) = sgn(I, J \cup K)sgn(J, K) = sgn(I \cup K, J)sgn(I, K)(-1)^{|J||K|}$ .  $\square$

Of course, multiplying by a sign, one can make this right action into a left action.

For any operad  $\mathcal{Q}$ , the unary operations  $\mathcal{Q}(1)$  form an algebra, hence in particular a Lie algebra, which acts on  $\mathcal{Q}$  by operadic derivations. Concretely, for  $q \in \mathcal{Q}(1)$ ,  $a \in \mathcal{Q}(n)$  the formula is

$$q \cdot a = c \circ_1 a - (-1)^{|a||c|} \sum_{j=1}^n a \circ_j q.$$

Suppose that in addition some Lie algebra  $\mathfrak{h}$  acts from the left on  $\mathcal{Q}$  by operadic derivations. Then also the Lie algebra  $\mathfrak{h} \ltimes \mathcal{Q}(1)$  acts on  $\mathcal{Q}$  by operadic derivations. Applying this to our case, we see that the Lie algebra

$$\hat{\mathfrak{g}} = \mathfrak{g} \ltimes \widetilde{Tw}\mathcal{P}(1)$$

acts on  $\widetilde{Tw}\mathcal{P}(1)$  by operadic derivations. Given some (homogeneous) element  $x \in \mathfrak{g}$ , we construct an element  $x_1 \in \widetilde{Tw}\mathcal{P}(1)$  by the formula

$$x_1(1, \bar{1}, \dots, \bar{j}) = x(1, \bar{1}, \dots, \bar{j}).$$

**Lemma 10.** *If  $\mu \in \mathfrak{g}$  is a Maurer-Cartan element, then  $\hat{\mu} = \mu - \mu_1$  is a Maurer-Cartan element in  $\hat{\mathfrak{g}} = \mathfrak{g} \ltimes \widetilde{Tw}\mathcal{P}(1)$ .*

*Proof.* Let us compute this for homogeneous  $\mu$ , the general case is analogous.

$$\begin{aligned}
\frac{1}{2} [\mu_1, \mu_1] (s_1, \bar{1}, \dots, \overline{2m-2}) &= \sum_{I \sqcup J = [2m-2]} \operatorname{sgn}(I, J)^{k+1} \mu(\mu(s_1, \bar{I}), \bar{J}) \\
&= - \sum_{I \sqcup J = [2m-2]} \operatorname{sgn}(I, J)^{k+1} (-1)^{m(k+1)} \mu(\mu(\bar{I}), s_1, \bar{J}) \\
&= - \sum_{I \sqcup J = [2m-2]} \operatorname{sgn}(I, J)^{k+1} (-1)^{k+1} \mu(s_1, \bar{J}, \mu(\bar{I})) \\
&= - \sum_{I \sqcup J = [2m-2]} \operatorname{sgn}(I, J)^{k+1} (-1)^{m(m-1)(k+1)} \mu(s_1, \bar{I}, \mu(\bar{J})) \\
&= -\mu_1 \cdot \mu(s_1, \bar{1}, \dots, \overline{2m-2}) = \mu \cdot \mu_1(s_1, \bar{1}, \dots, \overline{2m-2})
\end{aligned}$$

□

Given a Maurer-Cartan element in  $\hat{\mathbf{g}}$  we can twist the differential on  $\widetilde{Tw\mathcal{P}}$ .

**Definition 2.** Let  $\mathcal{P}$  be any operad and  $k \in \mathbb{Z}$  be an integer. Let  $\mu \in \operatorname{Def}(\operatorname{Lie}_\infty^{(k)} \rightarrow \mathcal{P})$  be a Maurer-Cartan element. Then we define the  $\mu$ -twisted operad  $Tw\mathcal{P}$  as the operad constructed above with differential

$$d_{\mathcal{P}} + \hat{\mu} \cdot.$$

By construction, one has an action on  $Tw\mathcal{P}$  by the  $\mu$ -twisted version of the Lie algebra  $\mathbf{g}$ , i.e., the Lie algebra  $\mathbf{g}$  with the term  $[\mu, \cdot]$  added to the differential.

**Remark.** Let  $\mathcal{P}, k, \mu$  be as above and suppose that a  $\mathcal{P}$ -algebra  $A$  is given. Let  $\mathbf{n}$  be a commutative nilpotent or pro-nilpotent algebra, e.g.,  $\mathbf{n} = \epsilon\mathbb{R}[[\epsilon]]$ . Let  $m \in A \otimes \mathbf{n}$  be a Maurer-Cartan element, i.e., a Maurer-Catan element in the  $hoLie_k$ -algebra  $A \otimes \mathbf{n}$ . Then  $A \otimes \mathbf{n}$  is an algebra over the operad  $Tw\mathcal{P}$  by the formula

$$p(x_1, \dots, x_n) = \frac{1}{j!} p(x_1, \dots, x_n, m, \dots, m)$$

for  $p \in Tw\mathcal{P}$  homogeneous wrt. the degree in  $j$  and  $a_1, \dots, a_n \in A$ .<sup>16</sup>

**G.3. More explicit description for the action on the Graphs operad.** Let us specialize the above constructions to the case  $\mathcal{P} = \operatorname{Gra}_n^\circ$ . Then  $\mathbf{g} = \operatorname{Def}(hoLie_{n-1} \rightarrow \mathcal{P}) = \mathbf{fGC}_n$  is the “full” graph complex, containing all possible graphs, possibly with multiple connected components, tadpoles or multiple edges. Elements of  $\mathbf{g}$  should be considered as (possibly infinite) linear combinations of graphs with numbered vertices, invariant under permutation of vertex labels. In pictures we draw a graph with black unlabelled vertices. This should be understood as the sum of all possible numberings of the vertices, divided by the order of the automorphism group. Note that the picture is still inaccurate since we do not specify the overall sign. An explicit Maurer-Cartan element  $\mu \in \mathbf{g}$  is given by the graph with two vertices and one edge.

By twisting one obtains the operad  $Tw\mathcal{P} = \mathbf{fGraphs}_n$ . The  $N$ -ary operations of  $Tw\mathcal{P}$  are (possibly infinite) linear combinations of graphs, having two kinds of numbered vertices, “internal” and “external”. It is required that there are exactly  $N$  external vertices and that the linear combination is invariant under interchange of the labels on the internal vertices. In pictures, we draw the internal vertices black without labels, with the convention that one should sum up over all possible numberings, and divide by the order of the symmetry group. From the Maurer-Cartan

<sup>16</sup>I admit that the notation here is suboptimal. The  $p$  on the left means the element of  $Tw\mathcal{P}$ , while the  $p$  on the right is the underlying operation in  $\mathcal{P}$ , which is (anti-)symmetric in its last  $j$  slots.

element  $\mu$  one obtains the element  $\mu_1 \in Tw\mathcal{P}(1)$ . It is given by the graph with one external and one internal vertex, and an edge between them. The differential on  $Tw\mathcal{P}$  has three terms: (i) There is a term coming from the action of  $\mu$  as in Lemma 9. Concretely, this amounts to splitting each internal vertex into two and reconnecting the incoming edges. (ii) There is a term  $\sum_j (\cdot) \circ_j \mu_1$ . This amounts to splitting off from each external vertex one internal vertex, and reconnecting the incoming edges. (iii) There is a term  $\mu_1 \circ_1 (\cdot)$ . This term adds a new internal vertex and connects it to every other vertex (but one at a time). Note that if all internal vertices are at least bivalent, the terms (iii) precisely cancel those terms from (i) and (ii) that contain graphs with univalent internal vertices.

Next consider more generally the action of an arbitrary element  $\gamma \in \mathfrak{g}$  on some  $\Gamma \in Tw\mathcal{P}$ . It again contains three terms: (i) There is a term coming from the action as in Lemma 9. It amounts to inserting  $\gamma$  at the internal vertices of  $\Gamma$  and reconnecting the incoming edges. (ii) Build the element  $\gamma_1 \in Tw\mathcal{P}(1)$  by marking the vertex 1 in  $\gamma$  as external. Then there is a term in the action stemming from  $\sum_j \Gamma \circ_j \mu_1$ . This amounts to inserting  $\gamma_1$  at all external vertices. (iii) There is the term  $\gamma_1 \circ_1 \Gamma$ . This amounts to inserting  $\Gamma$  at the external vertex of  $\gamma_1$ .

APPENDIX H. A DIRECT PROOF THAT THE TAMARKIN MAP  $\mathfrak{grt} \rightarrow H^0(\mathbf{GC}_2)$   
COINCIDES WITH THE ONE DESCRIBED BY THE SECOND  
ALGORITHM IN SECTION 6.

The argument in this subsection came out of a discussion with Pavol Ševera. As in section 4.3.1, we have a chain of quasi-isomorphisms of operads

$$(14) \quad CNPaCD \rightarrow BU\mathfrak{t} \leftarrow e_2 \leftarrow G_\infty.$$

Fix a lift up to homotopy  $G_\infty \rightarrow CNPaCD$  for now. It exists because  $G_\infty$  is cofibrant.

The Lie algebra  $\mathfrak{grt}$  acts on  $PaCD$ . Let some  $x \in \mathfrak{grt}$  be given. We have the following diagram (non-dashed arrows).

$$\begin{array}{ccccccc} L_\infty & \xrightarrow{\quad} & G_\infty & & & & \\ \downarrow & & \downarrow & \nearrow & & & \\ & & CNPaCD & & & & \\ & & \downarrow id + \epsilon x & & & & \\ G_\infty & \xrightarrow{\quad} & CNPaCD & \longrightarrow & \mathbf{Gra} & \longrightarrow & \text{End}(T_{\text{poly}}) \end{array}$$

Again by cofibrancy, one can find a lift (dashed arrow), up to a homotopy  $h$ . This (infinitesimal) automorphism of  $G_\infty$  describes the action of the  $\mathfrak{grt}$ -element  $x$  on  $G_\infty$ . The composition of the homotopy  $h$  with the maps  $L_\infty \rightarrow G_\infty$  from the left and  $CNPaCD \rightarrow \mathbf{Gra}$  from the left yields a degree 0 cocycle

$$\xi \in \text{Def}(L_\infty \rightarrow \mathbf{Gra})$$

which represents a cohomology class in the graph complex. Composing  $\xi$  with the map  $\mathbf{Gra} \rightarrow \text{End}(T_{\text{poly}})$  we recover D. Tamarkin's action of  $x$  on  $T_{\text{poly}}$ . The goal of this section is to show the following result:

**Proposition 26.** *The cocycle  $\xi$  describes the same graph cohomology class as the cocycle  $\Gamma$  produced by the Algorithm 2 in Section 6.*

We will show that the cochain  $\Gamma'$  produced in that algorithm agrees with  $\xi$  up to exact terms. First note that there are two gradings on  $CNPaCD$ , one by the (non-positive) degree and one coming from the grading by number of  $t_{ij}$ 's on  $\mathbf{t}$ . We will call the latter grading the  $t$ -grading. The differential on  $CNPaCD$ , call it  $d$ , has degree 1 wrt. the first and degree 0 wrt. the second grading. Precompose the map  $G_\infty \rightarrow CNPaCD$  we picked by  $L_\infty \rightarrow G_\infty$  to obtain a map  $L_\infty \rightarrow CNPaCD$ . It is defined by specifying the images of the generators, call them  $\mu_2, \mu_3, \dots$ . They have to satisfy equations of the form

$$(15) \quad d\mu_n = (\text{linear combination of } \mu_j \circ \mu_k \text{ for } j + k = n + 1.).$$

Furthermore, the degree of  $\mu_n$  must be  $3 - 2n$ . This means that  $\mu_n$  is some linear combination of chains of morphisms of  $PaCD$  of length  $2n - 3$ . We know that  $\mu_2 = t_{12}$ .

**Lemma 11.** *The map  $G_\infty \rightarrow CNPaCD$  lifting (14) up to homotopy may be chosen such that each  $\mu_n$  as above has degree  $n - 1$  with respect to the  $t$ -grading (the one by number of  $t_{ij}$ 's).*

*Proof.* We do an induction, using equation (15). Suppose  $\mu_2, \dots, \mu_{n-1}$  are as in the Lemma. Then the right hand side of (15) has  $t$ -degree  $n - 1$ . Hence the part of  $\mu_n$  of  $t$ -degree  $\neq n - 1$  must be closed. It follows that this part must be exact by (usual-)degree reasons (note that  $H(CNPaCD) \cong e_2$ ). Hence, by choosing a homotopic map  $G_\infty \rightarrow CNPaCD$  we can kill this part and proceed with the induction.  $\square$

**Remark.** In particular, this means that a chain of morphisms of  $PaCD$  occuring in  $\mu_n$  must contain at least  $2n - 3 - (n - 1) = n - 2$  morphisms of  $t$ -degree zero. In particular, mapping  $\mu_n$  along  $CNPaCD \rightarrow \mathbf{Gra}$  yields zero, except for  $n = 2$ .

Next consider the action of  $x \in \mathbf{grt}$ . It produces some

$$a \in \text{Def}(G_\infty \rightarrow CNPaCD)$$

of degree 1. Precompose with  $L_\infty \rightarrow G_\infty$  and compose with  $CNPaCD \rightarrow BU\mathbf{t}$  to obtain some element

$$a' \in \text{Def}(L_\infty \rightarrow BU\mathbf{t}).$$

By the last Remark one sees that  $a'$  is determined solely by the image of  $\mu_3$  under the action of  $x$ . By an explicit calculation one can see that  $a'$  in fact agrees with the element  $T_3 \in \text{Def}(L_\infty \rightarrow C\mathbf{t}) \subset \text{Def}(L_\infty \rightarrow BU\mathbf{t})$  from the second algorithm in section 4.3.1. Consider next the homotopy  $h$ . Precompose it with  $L_\infty \rightarrow G_\infty$  and compose with  $CNPaCD \rightarrow BU\mathbf{t}$  so as to obtain some degree 0 element

$$h' \in \text{Def}(L_\infty \rightarrow BU\mathbf{t}).$$

Because  $h$  was the homotopy making the lower right cell in the commutative diagram above commute, one has

$$a' = dh'$$

where  $d$  is now the differential in  $\text{Def}(L_\infty \rightarrow BU\mathbf{t})$ . Since  $a' = T_3$  actually lives in  $\text{Def}(L_\infty \rightarrow C\mathbf{t})$  and  $C\mathbf{t} \rightarrow BU\mathbf{t}$  is a quasi-isomorphism, there is a closed  $c \in \text{Def}(L_\infty \rightarrow BU\mathbf{t})$ , such that  $U := h' + c \in \text{Def}(L_\infty \rightarrow C\mathbf{t})$ . Since  $H^0(\text{Def}(L_\infty \rightarrow C\mathbf{t})) = 0$ ,  $c$  is in fact exact. Taking  $U$  for the  $U$  in the algorithm from section 4.3.1, we see that the output  $\gamma'$  of that algorithm is the image of  $U$  after composition with  $BU\mathbf{t} \rightarrow \mathbf{Gra}$ . The image of  $h'$  after composition with  $BU\mathbf{t} \rightarrow \mathbf{Gra}$  is  $\xi$ , hence

$$\gamma' = \xi + (\text{exact terms}).$$

$\square$

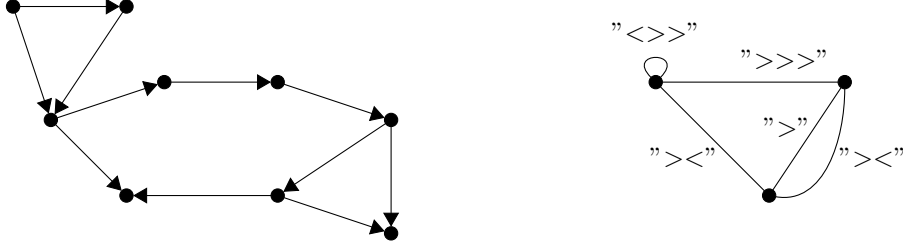


FIGURE 23. Passing from a graph with bivalent vertices to one with trivalent vertices only, but labelled edges. Note that we cheat a little, since one has to assign some directions, by some convention, to the edges on the right so as to interpret the labels correctly. However, for the argument in the proof this does not matter.

#### APPENDIX I. A SHORT NOTE ON THE DIRECTED VERSION OF THE GRAPH COMPLEX.

One may define a directed version of the graph complex  $\mathrm{GC}_n$  by (i) taking directed instead of undirected edges, (ii) allowing bivalent vertices and (iii) requiring that a graph has at least one trivalent vertex.<sup>17</sup> Call the resulting graph complexes  $\mathrm{dGC}_n$ .

**Proposition 27.**

$$H(\mathrm{dGC}_n) \cong H(\mathrm{GC}_n).$$

There is even an explicit quasi-isomorphism of dg Lie algebras

$$\mathrm{GC}_n \rightarrow \mathrm{dGC}_n$$

sending an undirected graph to a sum of directed graphs, obtained by interpreting each edge as the sum of edges in both directions.

*Proof of the Proposition.* Set up a spectral sequence on the number of bivalent vertices. The first differential produces one bivalent vertex. As in the undirected case, consider a graph with bivalent vertices as one with trivalent vertices and labelled edges, see Figure 23. The first differential just changes the labels. The resulting subcomplex is (essentially) a tensor product the complex for one edge. The cohomology of the latter can be seen to have a single nontrivial cohomology class, represented by the sum of edges in both directions (i.e.,  $\leftarrow + \rightarrow$ ). Hence the first convergent in the spectral sequence is  $\mathrm{GC}_n$  and hence  $\mathrm{GC}_n \rightarrow \mathrm{dGC}_n$  is a quasi-isomorphism.  $\square$

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<sup>17</sup>This last condition is to remove the loops as in Figure 5 and otherwise unnecessary.

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